The Natural Numbers. (Read pages 17-18.)

The symbol 0 denotes $\emptyset$. The successor function is $S(x) = x \cup \{x\}$. A set I is *inductive* if
(a) $0 \in I$, and
(b) $x \in I$ implies $S(x) \in I$.

The Axiom of Infinity asserts that there is an inductive set. Since there is at least one, the Axiom of Separation allows us to intersect all inductive sets. The resulting set is called "the natural numbers", and is denoted $\mathbb{N}$ or ω. (This is a definition: $\mathbb{N}$ is **defined** to be the intersection of all inductive sets.)

Theorem 1. *$\mathbb{N}$ is an inductive set.*

Proof. Let $\mathcal{I}$ be the (nonempty) class of all inductive sets. $\mathbb{N} = \bigcap \mathcal{I}$, that is, $\mathbb{N}$ is the set of elements common to all inductive sets.

Since 0 is common to all inductive sets, we get
(a) $0 \in \mathbb{N}$.

If $x \in \mathbb{N}$, then x is common to all inductive sets, so $S(x)$ is also common to all inductive sets. We get
(b) $x \in \mathbb{N}$ implies $S(x) \in \mathbb{N}$,

which completes the proof that $\mathbb{N}$ is inductive. $\square$

Since $\mathbb{N}$ is an inductive set that is a subset of every inductive set, it is often called "the least inductive set".

Corollary 2. *If I is an inductive set and $I \subseteq \mathbb{N}$, then $I = \mathbb{N}$.*

Recursion and Induction on $\mathbb{N}$.

Recursion is a technique for defining objects (functions, sets, ETC), while induction is a technique for proving statements about recursively-defined objects.

1. Recursion on $\mathbb{N}$.

The factorial function, $f(n) = n!$, is defined by recursion:

$$\begin{aligned} f(0) &:= 1 \\ f(S(k)) &:= S(k) \cdot f(k). \end{aligned}$$

This kind of definition ensures that the set $I = \{n \in \mathbb{N} \mid f(n) \text{ is defined}\}$ satisfies
(a) $0 \in I$, and
(b) $k \in I$ implies $S(k) \in I$.

By Corollary 2, $I = \mathbb{N}$, so $f(n)$ is defined for every natural number n.

For another example of a recursive definition, consider the definition of the *n-fold sum*, $\sum_{i=0}^{n} x_i$. The structure of the definition is:

$$\begin{aligned} \textstyle\sum_{i=0}^{0} x_i &:= x_0 \\ \textstyle\sum_{i=0}^{S(k)} x_i &:= \left(\textstyle\sum_{i=0}^{k} x_i\right) + x_{S(k)}. \end{aligned}$$

A general theorem concerning definition by recursion over $\mathbb{N}$ is

Theorem 3. *(The Recursion Theorem, version 1) Assume that $g : A \times \mathbb{N} \to A$ is a function, and that $a_0 \in A$. There is a unique function $f : \mathbb{N} \to A$ defined by*

$$\begin{aligned} f(0) &:= a_0 \\ f(S(k)) &:= g(f(k), k). \end{aligned}$$

2. Induction on $\mathbb{N}$.

When functions have been defined by recursion, we prove properties of those functions by a technique called *induction.* (So 'recursion' is the term used for definitions and 'induction' is the term used for proofs.)

Theorem 4. *(Induction is a valid form of proof, version 1) Assume that $s_0, s_1, \ldots$ is a sequence of statements indexed by the natural numbers. If*

(a) *s_0 is true, and*

(b) *$s_k \to s_{k+1}$ (that is, the truth of s_k implies the truth of s_{k+1}, for every k), then*

all s_n are true.

Proof. Let $I = \{n \in \mathbb{N} \mid s_n \text{ is true}\}$. Then (a) and (b) of the theorem imply that I is an inductive subset of $\mathbb{N}$, so $I = \mathbb{N}$ by Corollary 2. $\square$

Later we will prove laws of arithmetic for $\mathbb{N}$, and also prove that $S(k) = k + 1$. To write things in a less foreign-looking way, these things will be assumed in the following examples.

Example 1. Define $f(n)$ so that it equals the nth odd natural number. That is,

$$\begin{aligned} f(0) &:= 1 \\ f(k+1) &:= f(k) + 2. \end{aligned}$$

The function table for f, for small inputs, is

n	0	1	2	3	$\cdots$
$f(n)$	1	3	5	7	$\cdots$

After examining the values of $f(k)$ for small k we might make the conjecture that $f(k) = 2k+1$ for all k. To prove this by induction, we start by *writing down the exact statement to be proved, indexed by the natural numbers*:

$$s_k : f(k) = 2k+1.$$

A proof by induction involves two steps:

(a) (The basis of induction) Show s_0 is true.
(b) (The inductive step) Show that $s_k \to s_{k+1}$.

We write:

(Basis of induction) We must show that s_0 is true, that is that $f(0) = 2 \cdot 0 + 1 = 1$. This is the initial condition in the definition of f.

(Inductive step) We must show that s_k implies s_{k+1}. Assume that, for some k, we have $f(k) = 2k+1$ (i.e., s_k is true). We must show that $f(k+1) = 2(k+1)+1$ (i.e., s_{k+1} is true). Using the definition of f we calculate that

$$\begin{aligned} f(k+1) &= f(k) + 2 \\ &= (2k+1) + 2 \\ &= 2(k+1) + 1. \end{aligned}$$

This completes the proof.

Example 2. Define $f(n)$ so that it equals the sum of the first n odd natural numbers. That is,

$$\begin{aligned} f(0) &:= 1 \\ f(k+1) &:= f(k) + (2 \cdot (k+1) + 1). \end{aligned}$$

The function table for f, for small inputs, is

n	0	1	2	3	$\cdots$
$f(n)$	1	4	9	16	$\cdots$

We might make the conjecture that $f(k) = (k+1)^2$ for all k. To prove this by induction, we start by writing down the exact statement to be proved:

$$s_k : f(k) = (k+1)^2.$$

A proof by induction involves two steps:

(a) (The basis of induction) Show s_0 is true.
(b) (The inductive step) Show that $s_k \to s_{k+1}$.

We write:

(Basis of induction) We must show that s_0 is true, that is that $f(0) = 1^2 = 1$. This is the initial condition in the definition of f.

(Inductive step) We must show that s_k implies s_{k+1}. Assume that, for some k, we have $f(k) = (k+1)^2$ (i.e., s_k is true). We must show that $f(k+1) = ((k+1)+1)^2$ (i.e., s_{k+1} is true). Using the definition of f we calculate that

$$\begin{aligned} f(k+1) &= f(k) + (2 \cdot (k+1) + 1) \\ &= (k+1)^2 + (2 \cdot (k+1) + 1) \\ &= ((k+1)+1)^2. \end{aligned}$$

This completes the proof.

Test yourself! Use induction to prove these statements.

(i) s_n: $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$

(ii) s_n: $1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(iii) s_n: $1^3 + 2^3 + \cdots + n^3 = [\frac{n(n+1)}{2}]^2$

(iv) s_n: $1 + r + \cdots + r^n = \frac{r^{n+1}-1}{r-1}$ if $r \neq 1$.