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For an alternative proof, use the formula.

Pascal's Triangle

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$$
\begin{array}{c}
1 \\
1 \quad 1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1 \\
1 \quad 8 \quad 28 \quad 56 \quad 70 \quad 56 \quad 28 \quad 8 \quad 1 \\
1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1 \\
1 \quad 10 \quad 45 \quad 120 \quad 210 \quad 252 \quad 210 \quad 120 \quad 45 \quad 10 \quad 1
\end{array}
$$

Pascal's Triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
1 8 28 56 70 56 28 8 1
1 9 36 84 126 126 84 36 9 1
1 10 45 120 210 252 210 120 45 10 1

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Theorem. The nth row of Pascal's triangle is a symmetric, unimodal sequence that sums to 2^n.

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										1										
									1		1									
								1		2		1								
							1		3		3		1							
						1		4		6		4		1						
					1		5		10		10		5		1					
				1		6		15		20		15		6		1				
			1		7		21		35		35		21		7		1			
		1		8		28		56		70		56		28		8		1		
	1		9		36		84		126		126		84		36		9		1	
1		10		45		120		210		252		210		120		45		10		1

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					1		5		10		10		5		1					
				1		6		15		20		15		6		1				
			1		7		21		35		35		21		7		1			
		1		8		28		56		70		56		28		8		1		
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Unimodal, since $1 \le \binom{n}{k+1} / \binom{n}{k} = \frac{k!(n-k)!}{(k+1)!(n-k-1)!} = \frac{n-k}{k+1} \Leftrightarrow k \le \frac{n-1}{2}$.

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Theorem. $(x+y)^n = \binom{n}{0}x^n y^0 + \binom{n}{1}x^{n-1}y^1 + \binom{n}{2}x^{n-2}y^2 + \cdots + \binom{n}{n}x^0 y^n \quad (*)$.

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 & & + & \binom{n}{0}x^n y^1 & + & \cdots & + & \binom{n}{n-1}x^1 y^n & + \binom{n}{n}x^0 y^{n+1} \\
\hline
 & \binom{n+1}{0}x^{n+1}y^0 & + & \binom{n+1}{1}x^n y^1 & + & \cdots & + & \binom{n+1}{n}x^1 y^n & + \binom{n+1}{n+1}x^0 y^{n+1}
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- $\binom{n}{k_1,k_2,\ldots,k_r} = \binom{n-1}{k_1-1,k_2,\ldots,k_r} + \binom{n-1}{k_1,k_2-1,\ldots,k_r} + \cdots + \binom{n-1}{k_1,k_2,\ldots,k_r-1}$.

Pascal’s Pyramid

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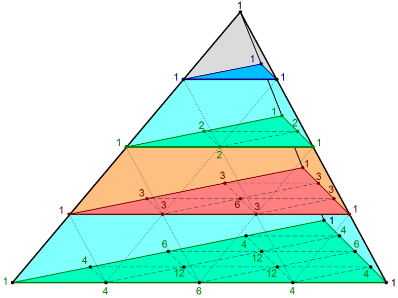
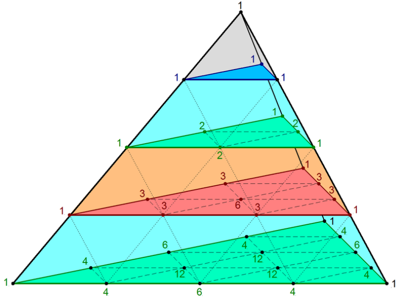
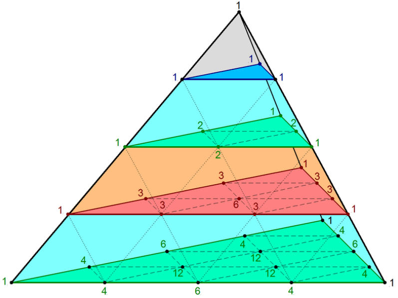




Multinomial Theorem.



Multinomial Theorem.

$(x_1 + x_2 + \cdots + x_r)^n = \sum_{k_1+k_2+\cdots+x_r=n} \binom{n}{k_1,k_2,\ldots,k_r} x_1^{k_1} x_2^{k_2} \cdots x_r^{k_r}.$

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"Stars and Bars" Proof.

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