The Principle of Inclusion and Exclusion.

Version 1. The principle of incl./exlc. counts the size of a union.

$$|A_1 \cup \cdots \cup A_n| = \sum_{1 \le i \le n} |A_i| - \sum_{1 \le i < j \le n} |A_i \cap A_j| + \cdots + (-1)^{n+1} |A_1 \cap \cdots \cap A_n|.$$

Version 2. Let X be a set and let $\mathcal{P}$ be a set of properties the elements of X may have. If $N_=(S)$ is the number of elements of X that have exactly the properties in $S \subseteq \mathcal{P}$ and $N_\geq(S)$ is the number of elements of X that have at least the properties in $S \subseteq \mathcal{P}$, then

$$\begin{aligned} N_\geq(S) &= \textstyle\sum_{S \subseteq T \subseteq \mathcal{P}} N_=(T) \quad \text{and} \\ N_=(S) &= \textstyle\sum_{S \subseteq T \subseteq \mathcal{P}} (-1)^{|T|-|S|} N_\geq(T). \end{aligned}$$

The first formula is trivial; the principle of inclusion and exclusion is the second formula.

Exercises.

(1) How many positive integers less than 1000 are not divisible by 2, 3, 5 or 7?

$1000 - 500 - 333 - 200 - 142 + 166 + 100 + 71 + 66 + 47 + 28 - 33 - 23 - 14 - 9 + 4 = 228.$

(2) How many positive integers less than 250 are not perfect powers?

It suffices to count the elements of $\{2, 3, \ldots, 250\}$ that are not squares, cubes, 5th powers or 7th powers. This is similar to the previous problem. The answer is 226.

(3) How many 5 digit numbers fail to contain the sequence 01? How about 00?

First part:

$$10^5 - 4 \cdot 10^3 + 3 \cdot 10 = 96030.$$

Second part:

$$10^5 - 4 \cdot 10^3 + 3 \cdot 10^2 + 3 \cdot 10 - 22 + 1 = 96309.$$

(4) How many 6 digit numbers have the property that, for every k, the kth digit is different than the $(7-k)$th digit?

$$\binom{3}{0} 10^6 - \binom{3}{1} 10^5 + \binom{3}{2} 10^4 - \binom{3}{3} 10^3 = 10^3 9^3 = 729000.$$