Finite versus Infinite

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which is also injective and which satisfies $\mathrm{im}(f') = \mathrm{im}(f)$.

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   Assume otherwise that there is a bijection between $\mathbb{N}$ and some $m \in \mathbb{N}$, If $f : \mathbb{N} \to m$ is a bijection (or even an injection), then for any $n > m$ we have that $f|_n : n \to m$ is injective. This contradicts the Pigeonhole Principle.

4. If $|\mathbb{N}| \leq |X|$, then X is infinite.

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Consequences of the Baby Pigeonhole Principle

1. (Pigeonhole Principle) If $n > m$, then there is no injective function $f: n \to m$. (Otherwise, $f: n \to n$ is injective and not surjective.)

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