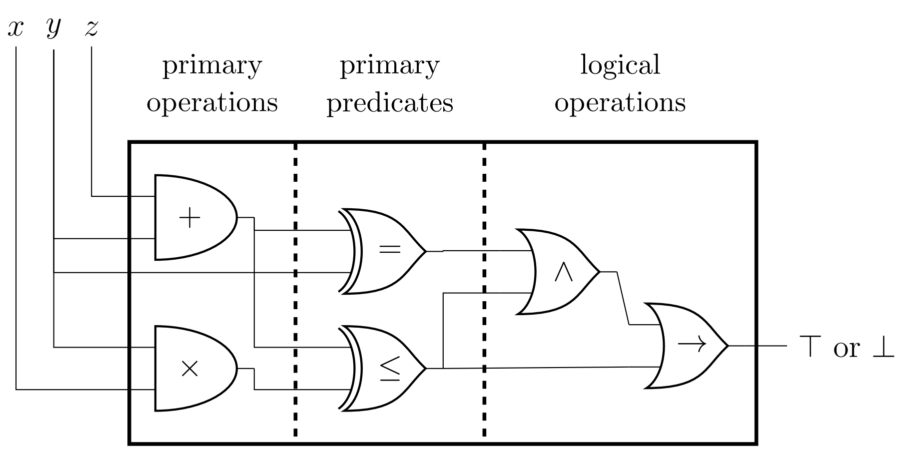
DISCRETE MATH (MATH 2001)

REVIEW SHEET II

Sections 2.7, 3.1, 3.2, 3.5, 3.6.1, 4.1–4.4, 4.6.1, 4.8, 6.1–6.3, 6.5, 6.6

IV. Cardinality.
- (a) Meaning of $|A| \leq |B|$, $|A| = |B|$, and $|A| < |B|$.
- (b) Finite and infinite. Countable and uncountable.
- (c) Pigeonhole Principle. $\mathbb{N}$ is infinite.
- (d) Cantor-Bernstein-Schroeder Theorem.
- (e) Cantor's Theorem.
- (f) $|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}| = |\mathbb{R}^n|$.

V. Logic.
- (a) Structures.
- (b) Alphabet of symbols. Ingredients in a compound predicate.



compound predicate

- (c) Deciding the truth of a statement in a structure.
  - (i) Assigning tables to terms.
  - (ii) Assigning tables to atomic formulas.
  - (iii) Logical connectives. Truth tables. Propositional tautology. Propositional contradiction. Logical equivalence of propositions. Logical implication of propositions. Logical independence of propositions.
  - (iv) Effect of logical connectives and quantifiers on the tables for predicates.
  - (vi) Disjunctive normal form.
  - (vii) Prenex form.
  - (viii) Quantifier games. Winning strategies.

(d) Proof.
  (i) Definition of "proof".
  (ii) Axioms. Logically valid sentence.
  (iii) Laws of deduction. Modus ponens, modus tollens.
  (iv) Direct proof, proof of the contrapositive, and proof by contradiction. Proof by cases.
  (v) The use of truth tables for designing proof strategies.
  (vi) The relationship between truth and provability: semantic consequence ($\Sigma \models S$) versus syntactic consequence ($\Sigma \vdash S$).
  (vii) Significance of Soundness and Completeness with regard to proof systems.
  (vii) Relevance of Gödel's Completeness Theorem.

VI. Counting.
(a) Additive counting principle and multiplicative counting principle.
(b) Number of functions $f : k \to n$. Characteristic functions. $|\mathcal{P}(n)|$.
(c) Number of injective functions $f : k \to n$.
(d) Number of bijective functions $f : k \to n$. Number of permutations of a finite set.
(e) If E is a uniform equivalence relation on X, then

$$|X/E| = |X|/(\text{common size of } E\text{-classes}).$$

(f) Binomial coefficients: definition, formula, recursion, Binomial Theorem, Pascal's Triangle, combinatorial proof.
(g) Multinomial coefficients: definition, formula, recursion, Multinomial Theorem, Pascal's Pyramid.
(h) Multichoose numbers: definition, formula.
(i) Inclusion-exclusion.
(j) Number of surjective functions $f : k \to n$.
(k) Stirling numbers of the second kind: definition, formula, recursion.

General advice on preparing for a math test.
Be prepared to demonstrate understanding in the following ways.

(i) Know the definitions of new concepts, and the meanings of the definitions.
(ii) Know the statements and meanings of the major theorems.
(iii) Know examples/counterexamples. (The purpose of an example is to illustrate the extent of a definition or theorem. The purpose of a counterexample is to indicate the limits of a definition or theorem.)
(iv) Know how to perform the different kinds of calculations discussed in class. (Here *calculation* refers to any routine or mechanical procedure, such as creating a truth table or putting a sentence in prenex form.)
(v) Be prepared to prove elementary statements. (Understanding the proofs done in class is the best preparation for this.)

(vi) Know how to correct mistakes made on old HW.

Is the final exam cumulative?

No. The final exam will cover only material discussed after the October 6 midterm.

Specifically, you will be examined only on Section 2.7 (Cardinality) and parts of Chapters 3, 4 (logic) and 6 (counting). But some of this later material relies on earlier material. For example, in some problems you might need to know what a subset is, even though that is not the focus of the problem. For another example, a question on logic might ask you to write a sentence formally. The sentence might be about sets or functions or natural numbers, and you might need to know about sets or functions or the natural numbers to write the sentence correctly.

Some definitions to know.

(1) Predicate. Operation. Structure.
(2) Logical connective. Truth table. Tautology. Contradiction.
(3) Contrapositive. Converse. Inverse.
(4) Disjunction. Conjunction. Disjunctive normal form.
(5) Proof. Axioms. Rules of deduction.
(6) Valid sentence.
(7) Semantic consequence. Syntactic consequence.
(8) Soundness. Completeness.
(9) Additive counting principal. Multiplicative counting principal.
(10) Binomial coefficient. Multinomial coefficient. Multichoose number. Stirling number.

Some theorems to know.

(1) Pigeonhole Principle.
(2) Cantor-Bernstein-Schroeder Theorem.
(3) Cantor's Theorem.
(4) Gödel's Completeness Theorem.
(5) Binomial Theorem. Multinomial Theorem.
(6) Inclusion-Exclusion Theorem.

Practice Problems.

(1) How do you answer a question where you are asked to "Give an example"?

Give an example of such a question.

(2) How do you answer a question where you are asked to "Give a definition"?

Define "definition".

(3) If you are asked to "Give a proof or counterexample", how do you decide which thing to do?

Give a proof or counterexample to the claim "Every prime is odd."

(4) Suppose the task is to define the word "function", and the answer given is "A function is something with an input and an output, like $f(x) = x+1$." Explain what is wrong with this answer.

(5) State the theorem.
(a) Cantor's Theorem.
(b) Cantor-Bernstein-Schroeder Theorem.

(6) True or False? Explain.
(a) If $A \subseteq B \subseteq C$ and $|A| = |C|$, then $|A| = |B|$.
(b) If A has an uncountable subset, then A is uncountable.

(7) Write a formal sentence expressing the Axiom of Union. Then draw a formula tree for your sentence.

(8) Suppose that P and Q are compound propositions.
(a) Must the implication $P \to Q$ be logically equivalent to its converse?
(b) Must the implication $P \to Q$ be logically inequivalent to its converse?
(c) Must the implication $P \to Q$ be logically independent of its converse?

(a) No. (Counterexample: let P and Q be propositional variables. If these do not look like 'compound' propositions, you could let $P = p \wedge p$ and $Q = q \wedge q$ where p and q are propositional variables.)
(b) No. (Counterexample: consider the case $P = Q$.)
(c) No. (Counterexample: consider the case $P = Q$.)

(9) A theorem with two hypotheses and one conclusion has the form $((H_1 \wedge H_2) \to C)$. Which of the following implications defines a valid proof strategy?
(a) $(H_1 \wedge (\neg C)) \to (\neg H_2)$ (the proof would look like $H_1, (\neg C), \cdots, (\neg H_2)$.)
(b) $((\neg C) \to ((\neg H_1) \wedge (\neg H_2)))$
(c) $(H_1 \to (H_2 \to C))$

Strategies (a) and (c) are valid, but (b) is not.

(10) A politician claims that if we adopt their plan, then everyone will benefit. Let $A =$ "we adopt their plan" and let $B(x) =$ "x benefits". This claim may be formalized as "$A \to (\forall x)B(x)$". When asked to justify their claim, the politician explains" "Imagine if we did not adopt the plan. Then X, Y, and Z are likely to happen, and some people will not benefit". This explanation may be formalized as "$(\neg A) \to (\exists x)(\neg B(x))$". Does this explanation justify the claim? Explain why or why not.

No, the explanation does not justify the claim. Rather, the explanation given justifies the INVERSE of the claim, and in general the inverse of an implication is logically independent of the implication.

(11) Write the following sentence in prenex form.

$$(\forall x)(\forall y)((x < y) \to ((\exists z)(z < x)) \wedge ((\exists z)((x < z) \wedge (z < y))) \wedge ((\exists z)(y < z)))$$

Decide the truth of this sentence in (a) $\langle \mathbb{R}; < \rangle$, (b) $\langle \mathbb{Z}; < \rangle$ by describing a winning strategy for the relevant quantifier.

(12) The following sentence expresses that the function $f(x) = 2x + 1$ is continuous at $x = 1$.

$$(\forall \varepsilon > 0)(\exists \delta > 0)(\forall x)((0 < |x - 1| < \delta) \to (|f(x) - f(1)| < \varepsilon)).$$

Explain why the following is a winning strategy for $\exists$ for this sentence in the structure $\langle \mathbb{R}; +, -, 0, \cdot, 1, |x|, <, f(x) \rangle$.

- $\forall$ chooses some ε.
- $\exists$ chooses $\delta = \varepsilon/2$.
- $\forall$ chooses some x.

Is $f(x) = 2x + 1$ continuous at $x = 1$?

The answer to the last question is 'Yes', since the strategy is a winning strategy for $\exists$.

Let's explain why the strategy is winning for $\exists$. If the chosen value for ε is ≤ 0, then $\exists$ has already won so any choice for δ is OK. In the case where $\varepsilon > 0$, the strategy tells $\exists$ to choose $\delta = \varepsilon/2$, and this will force $\delta > 0$, which we need. Next, $\forall$ chooses some x. If we do not have $(0 < |x - 1| < \delta)$, then $\exists$ has already won, so assume that $(0 < |x - 1| < \delta)$ holds. Focusing on the part of this that says $|x - 1| < \delta$ yields the following conclusions successively:

- $|x - 1| < \delta$, or
- $-\delta < x - 1 < \delta$, or
- $-\varepsilon/2 < x - 1 < \varepsilon/2$, or
- $-\varepsilon < 2x - 2 < \varepsilon$, or
- $-\varepsilon < (2x + 1) - 3 < \varepsilon$, or
- $-\varepsilon < f(x) - f(1) < \varepsilon$, or

- $|f(x) - f(1)| < \varepsilon$.

This shows that $\exists$ strategy is a winning one, i.e., following the strategy produces a choice for δ that satisfies the sentence.

(13) Prove that the empty set is finite.

We must show that there is a bijection between $\emptyset$ and some natural number. But $\emptyset = 0 \in \mathbb{N}$, so the identity function $\mathrm{id}_\emptyset \colon \emptyset \to 0$ is such a bijection.

(14) How many license plates have exactly 7 characters consisting of decimal digits $(0, 1, \ldots, 9)$ and letters of the alphabet $(a, b, \ldots, z)$? What if there are exactly 3 decimal digits, 4 letters of the alphabet, and the digits must come before the letters?

For the first part of the question, a license plate has 7 characters which can appear in any order and with any multiplicity, which are chosen from an alphabet of $10+26 = 36$ characters. The total will be $36 \cdot 36 \cdot \cdots \cdot 36 = 36^7$.

If we choose the first three characters from the decimal digits and the last four from the letters of the alphabet, a similar argument shows that the number of possible license plates is $10 \cdot 10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 10^3 \cdot 26^4$.

(15) How many license plates have 3 decimal digits and 4 letters of the alphabet if the plate must start and end with a digit, and either start with 9 or end with 9?

Let A be the set of plates that have 3 decimal digits and 4 letters and which start with 9 and end with a digit. Let B be the set of plates that have 3 decimal digits and 4 letters and which end with 9 and start with a digit. We need to compute $|A \cup B|$ $(= |A| + |B| - |A \cap B|)$.

First compute $|A|$. Each plate in A starts with 9, ends in a digit, and has one other digit in some position from 2-6. The number of these is

$$|A| = 1 \cdot \binom{10}{1} \cdot \binom{7-2}{1} \cdot \binom{10}{1} \cdot 26^4 = 500 \cdot 26^4.$$

(We choose the first character in 1 way, then we chose the last character in 10 ways, then we choose the location of the remaining digit in $\binom{7-2}{1}$ ways, then we choose which digit is the remaining one in $\binom{10}{1}$ ways. Finally we fill in the remaining 4 characters in 26^4 ways.)

One can see that $|B| = |A| = 500 \cdot 26^4$. We compute $|A \cap B|$ in exactly the same way. The only difference from what is above is that the first and last characters are BOTH 9, so one of the factors $\binom{10}{1} = 10$ should be $\binom{1}{1} = 1$. Thus $|A \cap B| = 50 \cdot 26^4$.

The final answer is $|A \cup B| = 500 \cdot 26^4 + 500 \cdot 26^4 - 50 \cdot 26^4 = 950 \cdot 26^4 = 434127200$.

(16) How many ways are there to distribute 12 different books to 3 people? What if each person must get at least one book?

See the solution to Problem 1 of the December 1 handout.

(17) How many ways are there to distribute 12 identical textbooks to three shelves? How many ways to distribute 12 different books to three shelves? (In the second question, assume that the order of the books on each shelf matters.)

See the solution to Problem 2 of the December 1 handout.

(18) How many positive integral solutions are there to the equation

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 = 100?$$

How many nonnegative integral solutions are there?

See the solution to Problem 4 of the December 1 handout.

(19) How many ways are there to make 3 fruit baskets from 8 pineapples, 10 pomegranates, 6 coconuts and 20 figs if each basket must contain each kind of fruit?

See the solution to Problem 5 of the December 1 handout.

(20) In your department, everyone works in one of three workgroups. Workgroups A and B each have a total of 23 workers; Workgroup C has a total of 32 workers; there are 5 workers who belong in both groups A and B; 10 workers who belong in both B and C; 8 workers who belong in both A and C; and three workers who belong in all three groups. How many workers are in your department?

We must calculate $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$. We are given that $|A| = |B| = 23$, $|C| = 32$, $|A \cap B| = 5$, $|B \cap C| = 10$, $|A \cap C| = 8$, and $|A \cap B \cap C| = 3$. Adding these number with the appropriate sign yields $|A \cup B \cup C| = 58$.

(21) How many positive integers less than 1000 are not divisible by 2, 3, 5 or 7?

See the solution to Problem 1 of the December 6 handout.

(22) How many 5 digit numbers fail to contain the sequence 01? How about 00?

See the solution to Problem 3 of the December 6 handout.

(23) How many 6 digit numbers have the property that, for every k, the kth digit is different than the $(7-k)$th digit?

See the solution to Problem 4 of the December 6 handout.

(24) Give a combinatorial proof that $S(n, n-1) = C(n, 2)$.

Let A be the set of partitions of $X = \{x_1, \ldots, x_n\}$ into $n-1$ cells and let B be the set of 2-element subsets of X. A typical member of A will be a partition in which all but one of the cells are singletons, and the final cell is a doubleton. Such a partition might be written $\{\{x_1, x_2\}, \{x_3\}, \ldots \{x_n\}\}$ or more informally as $x_1x_2/x_3/\cdots/x_n$. A typical member of B might be written as $\{x_i, x_j\}$ where $1 \leq i < j \leq n$.

According to the definitions of the functions $S(n, k)$ and $C(n, k)$, $|A| = S(n, n-1)$ and $|B| = C(n, 2)$. To answer the question, it will suffice to exhibit a bijection $f \colon B \to A$. Informally, we might describe f as

$$\{x_i, x_j\} \mapsto x_ix_j/x_1/\cdots/x_n.$$

More formally, define f so that, for $\{x_i, x_j\} \in B$, $f(\{x_i, x_j\})$ is the partition of X which has a single cell $\{x_i, x_j\}$ that is a doubleton and all other cells are singletons. This is a partition of X into $n-1$ cells, so $f(\{x_i, x_j\}) \in A$. It is easy to see how to invert f: $f^{-1} \colon A \to B$ is defined so that if $\Pi \in A$, then $f(\Pi) =$ the 2-element cell of Π.

The fact that f is a bijection from B to A establishes that $S(n, n-1) = |A| = |B| = C(n, 2)$.

(25) Suppose that $|A| = n$ and $|B| = m$. How many pairs (X, Y) are there where $X = \mathrm{coim}(f)$ and $Y = \mathrm{im}(f)$ for some function $f \colon A \to B$?

Recall that if $f \colon A \to B$, then the induced function $\overline{f} \colon \mathrm{coim}(f) \to \mathrm{im}(f)$ is a bijection. This implies that $|\mathrm{coim}(f)| = |\mathrm{im}(f)|$; let k denote the common size of $\mathrm{coim}(f)$ and $\mathrm{im}(f)$. Since $\nu \colon A \to \mathrm{coim}(f)$ is a surjection we have $n = |A| \geq |\mathrm{coim}(f)| = k$, and similarly since $\iota \colon \mathrm{im}(f) \to B$ is an injection we have $k = |\mathrm{coim}(f)| \leq |B| = m$. Thus, $0 \leq k \leq \min(n, m)$. There is no other restriction on k.

The number of choices for $X = \mathrm{coim}(f)$, for a given choice of k, is $S(n, k)$ (= the number of partitions of A into k cells). The number of choices for $Y = \mathrm{im}(f)$, for the same choice of k, is $C(m, k)$ (= the number of subsets of B of size k). The number of pairs (X, Y), for this fixed k, is $S(n, k) \cdot C(m, k)$. Since k may vary from 0 to $\min(n, m)$, the Additive Sum Principle yields the final answer: $\sum_{k=0}^{\min(n,m)} S(n, k) \cdot C(m, k)$.