Solutions to HW 9.

1. This problem involves a deck of 52 distinct playing cards.
   (a) In how many ways can a 13-card bridge hand be dealt from the deck?

   We want an ordered sequence of 13 cards chosen from a 52-element set. There are $13! \cdot \binom{52}{13} = (52)_{13} = \frac{52!}{39!}$ of these.

   (b) How many different 13-card bridge hands are there?

   We want a 13-card subset of a 52-element set. There are $\binom{52}{13} = \frac{52!}{13!39!}$ of these.

2. (a) How many paths are there from the point $(0, 0)$ of $\mathbb{R}^2$ to the point $(10, 15)$ of $\mathbb{R}^2$ if each path consists of a sequence of steps of length 1 moving in the direction of the positive x-axis or the positive y-axis?

   We can describe a path by a list of instructions of the form "$(x, x, y, x, y, y, \ldots, x, y, y)$", which is a string of 10 x's and 15 y's in some order. If the first two instructions are x, then we take our first two steps in the x direction; if our next instruction is y, we take our next step in the y direction. ETC.
   The number of paths is equal to the number of lists of instructions, which is equal to the number of strings of length $10 + 15 = 25$ consisting of x's and y's, which contain 10 x's and 15 y's. This number is $\binom{25}{10} = \frac{25}{10!15!}$. (You have 25 instructions, and you "choose" 10 instructions to be x's and let the remaining 15 instructions be y's.)

   (b) How many paths are there from the point $(0, 0, 0)$ of $\mathbb{R}^3$ to the point $(10, 15, 20)$ of $\mathbb{R}^3$ if each path consists of a sequence of steps of length 1 moving in the direction of the positive x-axis, the positive y-axis or the positive z-axis?

   Using the same reasoning, we want to count strings of length $10 + 15 + 20 = 45$ which have 10 x's, 15 y's, and 20 z's. The number is $\binom{45}{10,15,20} = \frac{45!}{10!15!20!}$.

3. Let $MC(n, k)$ be the number "n-multichoose-k". Use a combinatorial argument to show that $MC(n, 0) + MC(n, 1) + \cdots + MC(n, k) = MC(n + 1, k)$.

Solution 1. (A combinatorial argument.) Let D be the set of all distributions of k identical balls to $n + 1$ distinct boxes with repetition allowed. $|D| = MC(n + 1, k) = \left(\binom{n+1}{k}\right) = \binom{n+k}{k}$.

Partition D into sets $D_0, D_1, \ldots, D_k$ where $D_i \subseteq D$ is the number of distributions where i balls are distributed to the first n boxes, while the remaining $k - i$ balls are distributed to the last box. Since we can distribute i identical balls to the first n boxes in $MC(n, i)$ ways, and we have no choice but to put the remaining $k - i$ balls into the last box, we have $|D_i| = MC(n, i)$. Since we have a partition,

$$MC(n + 1, k) = |D| = |D_0| + |D_1| + \cdots + |D_k| = MC(n, 0) + MC(n, 1) + \cdots + MC(n, k).$$

Solution 2. (Not a combinatorial argument, but OK.)

$$\begin{aligned}
MC(n,0) + MC(n,1) + \cdots + MC(n,k) &= \binom{n-1}{0} + \binom{n}{1} + \binom{n+1}{2} + \cdots + \binom{n+k-1}{k} \\
&= \left[\binom{n}{0} + \binom{n}{1}\right] + \binom{n+1}{2} + \cdots + \binom{n+k-1}{k} \\
&= \left[\binom{n+1}{1}\right] + \binom{n+1}{2} + \cdots + \binom{n+k-1}{k} \\
&= \left[\binom{n+1}{1} + \binom{n+1}{2}\right] + \cdots + \binom{n+k-1}{k} \\
&\quad \vdots \\
&= \binom{n+k-1}{k-1} + \binom{n+k-1}{k} \\
&= \binom{n+k}{k} = MC(n+1,k).
\end{aligned}$$