**Solutions to HW 8.**

1. Show that "$A \subseteq B$ and $B \subseteq A$ implies $A = B$" in each of the following two ways.

(a) With a direct proof.

We assume that $(A \subseteq B) \wedge (B \subseteq A)$, and argue that $A = B$.

Choose any $a \in A$. Since $A \subseteq B$, it follows that $a \in B$. Now choose any $b \in B$. Since $B \subseteq A$, it follows that $b \in A$. We have shown that $A$ and $B$ have the same elements, so $A = B$ by the Axiom of Extensionality.

(b) With a proof of the contrapositive.

Assume that $A \neq B$. The two sets do not have the same elements, so either (i) there is some $x$ in $A$ that is not in $B$ ($x \in A - B$) or (ii) there is some $y$ in $B$ that is not in $A$ ($y \in B - A$). In Case (i), $A \not\subseteq B$, while in Case (ii), $B \not\subseteq A$. This shows that

$$(A \neq B) \to (A \not\subseteq B) \vee (B \not\subseteq A),$$

which is the contrapositive statement.

2. Prove the statement "If $A \cap B = \emptyset$ and $A \cup B = B$, then $A = \emptyset$" in each of the following two ways.

(a) With a direct proof.

Assume that $A \cap B = \emptyset$ and $A \cup B = B$. We have

$$\begin{aligned} \emptyset &= A \cap \underline{B} && \text{Assumption} \\ &= A \cap (\underline{A \cup B}) && \text{Assumption} \\ &= A && \text{Absorption Law, (true of any sets } A, B\text{)} \end{aligned}$$

A second direct proof:

$$\begin{aligned} \emptyset &= A \cap \underline{B} && \text{Assumption} \\ &= A \cap (\underline{A \cup B}) && \text{Assumption} \\ &= (A \cap A) \cup (A \cap B) && \text{Distributive Law} \\ &= (A \cap A) \cup \emptyset && \text{Assumption} \\ &= A \cap A && \emptyset \text{ is a unit element for } \cup \\ &= A && \text{Idempotence of } \cap \end{aligned}$$

(b) With a proof by contradiction.

Assume that $A \cap B = \emptyset$, $A \cup B = B$, but $A \neq \emptyset$. Since $A \neq \emptyset$, there exists some $x \in A$. From the definition of union, $x \in A \cup B$ ($= B$). Therefore $x \in A$ and $x \in B$, so $x \in A \cap B$, contradicting $A \cap B = \emptyset$.

3. The goal of this problem is to prove that the composition of two injective functions is injective. The type of structure involved looks like $\mathbb{X} = \langle A, B, C; f, g\rangle$ where $f : A \to B$ and $g : B \to C$ are functions. Let the variables $a, a'$ range over the set $A$ and and the variables $b, b'$ range over the set $B$.

   The functions (i) $f$, (ii) $g$, (iii) $g \circ f$ are injective if the following sentences hold in $\mathbb{X}$:

   (i) $(\forall a)(\forall a')((f(a) = f(a')) \to (a = a'))$.

   (ii) $(\forall b)(\forall b')((g(b) = g(b') \to (b = b'))$.

   (iii) $(\forall a)(\forall a')((g \circ f(a) = g \circ f(a')) \to (a = a'))$.

   To prove that the composition of injective functions is injective, you must give a winning strategy for $\exists$ in the sentence in (iii). YOU ARE ALLOWED TO USE the fact that there exist winning strategies for $\exists$ in the sentences in (i) and (ii). Write a proof that indicates the winning strategy for $\exists$ in (iii), which accesses the information of the strategies for $\exists$ in (i) and (ii).

We must provide a winning strategy for $\exists$ for Game (iii). We are allowed to use that there exist winning strategies for $\exists$ in Games (i) and (ii).

Since the existential quantifier $\exists$ does not appear in any of the sentences, the only possible strategy for $\exists$ in any of these games is “Do nothing”. That is, the strategy for $\exists$ is “just watch while $\forall$ plays”. We need to explain why this is a winning strategy for $\exists$ in Game (iii).

Each of the sentences has the form

$$(\text{Quantifiers})(\text{Premise} \to \text{Conclusion}).$$

Recall that $P \to C$ fails only when the premise $P$ is true and the conclusion $C$ is false. We will use this fact.

We argue that “Do nothing” is a winning strategy for $\exists$ for Game (iii). Suppose

- (Play 1): $\forall$ chooses some $a \in A$.
- (Play 2): $\forall$ chooses some $a' \in A$.

We must show that $(g \circ f(a) = g \circ f(a')) \to (a = a')$ holds.

**Case 1.** (The premise $P$: “$g \circ f(a) = g \circ f(a')$” of sentence (iii) does not hold.)

In this case, $\exists$ has won, since $(P \to C)$ is true when the premise does not hold.

**Case 2.** (The premise $P$: “$g \circ f(a) = g \circ f(a')$” of sentence (iii) DOES hold.)

In this case, let $b = f(a)$ and $b' = f(a')$. (Here $\exists$ is imagining an instance of Game (ii) where $\forall$ chooses $b = f(a)$ and $b' = f(a')$.) Since the premise of sentence (iii) holds, we obtain $g(b) = g(f(a)) = g(f(a')) = g(b')$, establishing that the premise of sentence (ii) holds. Since we are assuming that the sentence in (ii) is true, the conclusion $b = b'$ of Game (ii) holds. Now, $f(a) = b = b' = f(a')$, establishing that the premise of statement (i) holds. Since we are assuming that the sentence in (i) is true, the conclusion $a = a'$ holds. This establishes that $\exists$ wins in Case 2.

These are the only cases, so “Do nothing” is a winning strategy for $\exists$.