Solutions to HW 6.

1. Show that $f\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}\colon (m,n)\mapsto 2^m(2n+1)-1$ is a bijection. (This shows that $|\mathbb{N}\times\mathbb{N}| = |\mathbb{N}|$ using a different argument than the one given in class.)

(f is injective) If $f(m,n) = f(p,q)$, then $2^m(2n+1)-1 = 2^p(2q+1)-1$, or $2^m(2n+1) = 2^p(2q+1)$ (a nonzero natural number). By uniqueness of prime factorizations, every nonzero natural number is expressible in a unique way as a product of a power of 2 and an odd number, hence $2^m = 2^p$ and $2n+1 = 2q+1$. From $2^m = 2^p$ we derive $m = p$ by unique factorization, and from $2n+1 = 2q+1$ we derive $2n = 2q$, and then $n = q$ with some arithmetic. This shows that $f(m,n) = f(p,q)$ implies $(m,n) = (p,q)$.

(f is surjective) If $k\in\mathbb{N}$, then we can solve $2^x(2y+1)-1 = k$ for $x, y\in\mathbb{N}$. Simply write this as as $2^x(2y+1) = k+1$, then choose x so that 2^x is the exact power of 2 that divides $k+1$ and choose y so that $2y+1$ is the odd number that remains after dividing $k+1$ by 2^x. The statement that $2^x(2y+1)-1 = k$ is solvable for $x, y\in\mathbb{N}$ is exactly what it means for f to be surjective.

2. Show that if $|X| = |Y|$, then $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$.

I will write the solution in a general form below, but first let me explain the idea of the solution in small example. Let $X = \{0,1\}$ and $Y = \{a,b\}$. Let $f\colon X\to Y$ be the bijection $0\mapsto a, 1\mapsto b$. Think of f as "renaming the elements $0,1$ using a,b". To prove that $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$ we need to establish that there is a bijection $\hat{f}\colon \mathcal{P}(X)\to\mathcal{P}(Y)$. Written out more fully, we need to establish that there is a bijection

$$\hat{f}\colon \{\emptyset, \{0\}, \{1\}, \{0,1\}\}\to\{\emptyset, \{a\}, \{b\}, \{a,b\}\}.$$

The idea is to take $\hat{f}$ to be the function that "renames inside the braces". That is $\hat{f}(\emptyset) = \emptyset$, $\hat{f}(\{0\}) = \{a\}$, $\hat{f}(\{1\}) = \{b\}$, $\hat{f}(\{0,1\}) = \{a,b\}$. So, the idea $\hat{f}(\{0,1\}) = \{f(0), f(1)\}$ may be expressed as "renaming a set with $\hat{f}$ means renaming the elements inside the set with f".

Let's write this down for general X and Y. Since $|X| = |Y|$, there is a bijection $f\colon X\to Y$. Let $g\colon Y\to X$ be the inverse of f. Define functions $\hat{f}\colon \mathcal{P}(X)\to\mathcal{P}(Y)$ by the rule $\hat{f}(S) = \{f(x)\in Y \mid x\in S\}$ and $\hat{g}\colon \mathcal{P}(Y)\to\mathcal{P}(X)$ by the rule $\hat{g}(T) = \{g(y)\in X \mid y\in T\}$. It is not hard to see that $\widehat{g}$ is the inverse of $\widehat{f}$, as follows:

$$\widehat{g}\circ\widehat{f}(S) = \widehat{g}(\{f(x)\in Y \mid x\in S\}) = \{g\circ f(x)\in X \mid x\in S\} = \{x\in X \mid x\in S\} = S$$

for any $S\in\mathcal{P}(X)$, and a similar argument shows that $\widehat{f}\circ\widehat{g}(T) = T$ for any $T\in\mathcal{P}(Y)$.

Since $\hat{f}\colon \mathcal{P}(X)\to\mathcal{P}(Y)$ is invertible, it is a bijection, hence $|\mathcal{P}(X)| = |\mathcal{P}(Y)|$.

3. Let $\mathrm{Eq}(\mathbb{N})$ be the set of equivalence relations on $\mathbb{N}$. Show that $|\mathcal{P}(\mathbb{N})| \leq |\mathrm{Eq}(\mathbb{N})| \leq |\mathcal{P}(\mathbb{N} \times \mathbb{N})|$. Use Problem 2 to conclude that $|\mathrm{Eq}(\mathbb{N})| = |\mathcal{P}(\mathbb{N})|$.

Let $\mathbb{N}^+$ be the set of nonzero natural numbers. According to the "Laws of Successor" (Sep 20 handout "arithmetic.pdf"), the successor function $S\colon \mathbb{N} \to \mathbb{N}^+$ is a bijection, so $|\mathbb{N}| = |\mathbb{N}^+|$. By Problem 2, $|\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{N}^+)|$.

Now we define a function $f\colon \mathcal{P}(\mathbb{N}^+) \to \mathrm{Eq}(\mathbb{N})$ by defining $f(S)$ to equal the equivalence relation on $\mathbb{N}$ with one equivalence class equal to $S \cup \{0\}$ and all other equivalence classes equal to singletons. That is, for $S \in \mathcal{P}(\mathbb{N}^+)$, define $f(S) = (S \cup \{0\})^2 \cup \{\{(n, n)\} \mid n \in \mathbb{N}\}$ The function f is injective, since if $S, T \in \mathcal{P}(\mathbb{N}^+)$ and $f(S) = f(T) = E$, then the E-equivalence class containing 0 is $S \cup \{0\} = T \cup \{0\}$; since both S and T consist of nonzero elements, $S = T$. Altogether this shows that

$$|\mathcal{P}(\mathbb{N})| = |\mathcal{P}(\mathbb{N}^+)| \leq |\mathrm{Eq}(\mathbb{N})|. \tag{1}$$

Next, every equivalence relation on $\mathbb{N}$ is a binary relation on $\mathbb{N}$, hence is a subset of $\mathbb{N} \times \mathbb{N}$. This means that $\mathrm{Eq}(\mathbb{N}) \subseteq \mathcal{P}(\mathbb{N} \times \mathbb{N})$, hence $|\mathrm{Eq}(\mathbb{N})| \leq |\mathcal{P}(\mathbb{N} \times \mathbb{N})|$. Since $|\mathbb{N} \times \mathbb{N}| = |\mathbb{N}|$, we derive from Problem 2 that

$$|\mathrm{Eq}(\mathbb{N})| \leq |\mathcal{P}(\mathbb{N} \times \mathbb{N})| = |\mathcal{P}(\mathbb{N})|. \tag{2}$$

Combining (1) and (2) yields $|\mathcal{P}(\mathbb{N})| \leq |\mathrm{Eq}(\mathbb{N})| \leq |\mathcal{P}(\mathbb{N})|$, so $|\mathrm{Eq}(\mathbb{N})| = |\mathcal{P}(\mathbb{N})|$ by the Cantor-Bernstein-Shröder Theorem.