Solutions to HW 3.

1. Explain why it is true that the function $F : A \to \mathcal{P}(A) : a \mapsto \{a\}$ is injective.

If $F(a) = F(b)$, then $\{a\} = \{b\}$, according to the definition of F. Now, by the Axiom of Extensionality, $a = b$. This establishes that F is injective. (We showed that $F(a) = F(b)$ implies $a = b$.)

2. In this problem, $f : A \to B$ and $g : B \to C$ will be composable functions.

(a) Show that if $g \circ f$ is injective, then f is injective, while if $g \circ f$ is surjective, then g is surjective.

Assume that $f(a_1) = f(a_2)$. Since g is a function, $g(f(a_1)) = g(f(a_2))$. This may also be written as $g \circ f(a_1) = g \circ f(a_2)$. By the injectivity of $g \circ f$, we get that $a_1 = a_2$. Altogether, this shows that $f(a_1) = f(a_2)$ implies $a_1 = a_2$, so f is injective.

(b) Show that if $g \circ f$ is surjective, then g is surjective.

Assume that $g \circ f : A \to C$ is surjective. It is our goal to prove that $g : B \to C$ is surjective. To this end, choose $c \in C$ arbitrarily; our aim is to show that there exists $b \in B$ such that $g(b) = c$. Since $g \circ f : A \to C$ is surjective, there is some $a \in A$ such that $c = g \circ f(a) = g(f(a))$. If we take $b = f(a)$, then we obtain that $g(b) = g(f(a)) = c$, as desired.

3. This is a continuation of Problem 2, so assume that $f\colon A \to B$ and $g\colon B \to C$ are composable functions.

(a) Give an example where $g \circ f$ is injective, but g is not injective.

(b) Give an example where $g \circ f$ is surjective but f is not surjective.

We give an example that works for both Part (a) and Part (b).

Let $A = \{0\} = C$ and let $B = \{0, 1\}$. Let $f : A \to B$ be defined so that $f(0) = 0$, and let $g : B \to C$ be defined so that $g(0) = g(1) = 0$. Then $g \circ f : A \to C$ is the identity function, so it is both injective and surjective. But f is not surjective and g is not injective.