Solutions to HW 2.

1. We have explained why the Russell class $\mathcal{R} = \{x \mid x \notin x\}$ is not a set. Explain why the following classes are also not sets.
   (a) The class $\mathcal{C}$ of all sets.
   (b) The class $\mathcal{D}$ of all 1-element sets.
   (Hints: For (a), show that the assumption that $\mathcal{C}$ is a set allows you to construct $\mathcal{R}$ as a set. For (b), show that the assumption that $\mathcal{D}$ is a set allows you to construct $\mathcal{C}$ as a set.)

   (a) The class $\mathcal{C}$ of all sets.

   If $\mathcal{C}$ was a set, then we could construct $\mathcal{R} = \{x \in \mathcal{C} \mid x \notin x\}$ with the Axiom of Separation, which is a legal set-theoretic construction. Thus, "$\mathcal{C}$ is a set" implies "$\mathcal{R}$ is a set". Since we know "$\mathcal{R}$ is not a set", then we must conclude that "$\mathcal{C}$ is not a set".

   (b) The class $\mathcal{D}$ of all 1-element sets.

   If $\mathcal{D} = \{\{A\}, \{B\}, \ldots\}$ were a set, then $\bigcup \mathcal{D} = \{A, B, \ldots\} = \mathcal{C}$ would be a set. We saw in Part (a), that $\mathcal{C}$ is not a set, so we conclude that $\mathcal{D}$ cannot be a set, either.

2. Your friend offers a wager that, under the Kuratowski encoding, the ordered pair $(0, 1)$ equals the natural number three. Should you take the wager? Explain.

Take the wager! (By offering the wager, the friend bets that the statement is true. We argue that it is false, so your friend will lose and you will win.)

$0 = \{\}, 1 = \{0\}, 2 = \{0, 1\}$ and $3 = \{0, 1, 2\}$. Thus $(0, 1) = \{\{0\}, \{0, 1\}\} = \{1, 2\} \neq \{0, 1, 2\} = 3$. (We know that $\{1, 2\} \neq \{0, 1, 2\}$ since only one of these two sets has an element equal to the empty set.)

(Here is a second solution: any ordered pair (x, y), considered as a set $\{\{x\}, \{x, y\}\}$, contains one or two distinct elements. But 3 contains three distinct elements, so $(x, y) \neq 3$ for any x and y.)

3. Show that $\emptyset \times A = \emptyset$.

We must show that $\emptyset \times A$ has no elements. We argue by contradiction: Assume that $\emptyset \times A$ has an element x. Then $x \in \emptyset \times A$ must have the form of an ordered pair, $x = (y, z)$, where $y \in \emptyset$ and $z \in A$. But $y \in \emptyset$ is impossible, since $\emptyset$ has no elements. This contradicts our assumption that $\emptyset \times A$ has an element. Since the assumption that $\emptyset \times A$ has an element has been shown to be false, we derive that $\emptyset \times A = \emptyset$.