1. How many 5-card poker hands have cards of every suit?

To count the number of hands that have cards of every suit, we will use inclusion/exclusion.

Let H be the set of all 5-card hands. Let $H_\diamondsuit$ be the set of 5-card hands that have no $\diamondsuit$'s, let $H_\clubsuit$ be the set of 5-card hands that have no $\clubsuit$'s, let $H_\spadesuit$ be the set of 5-card hands that have no $\spadesuit$'s, and let $H_\heartsuit$ be the set of 5-card hands that have no $\heartsuit$'s. Our goal is to calculate

$$|H| - |H_\diamondsuit \cup H_\clubsuit \cup H_\spadesuit \cup H_\heartsuit|.$$

It is easy to see that

- $|H| = \binom{52}{5}$.
- $|H_\diamondsuit| = \binom{52-13}{5} = \binom{39}{5}$ $(= |H_\clubsuit| = |H_\spadesuit| = |H_\heartsuit|)$.
- $|H_\diamondsuit \cap H_\clubsuit| = \binom{52-13-13}{5} = \binom{26}{5}$.
- $|H_\diamondsuit \cap H_\clubsuit \cap H_\spadesuit| = \binom{52-13-13-13}{5} = \binom{13}{5}$.
- $|H_\diamondsuit \cup H_\clubsuit \cap H_\spadesuit \cap H_\heartsuit| = \binom{0}{5}$.

By inclusion/exclusion

$$|H| - |H_\diamondsuit \cup H_\clubsuit \cup H_\spadesuit \cup H_\heartsuit| = \binom{52}{5} - \left(\binom{4}{1}\binom{39}{5} - \binom{4}{2}\binom{26}{5} + \binom{4}{3}\binom{13}{5} - \binom{4}{4}\binom{0}{5}\right) = 685464.$$

Alternative solution: The number of hands of cards with the desired properties could be counted as follows. Each hand must have one card of every suit, except that the hand will have two cards from one of the suits which I will call the 'doubled' suit. Compute as follows: Choose which suit is the doubled suit (this can be done in $\binom{4}{1}$ ways), choose the two numbers of the doubled suit (this can be done in $\binom{13}{2}$ ways), then choose one number for each of the other suits (this can be done in $\binom{13}{1}$ ways for each one of the other suits). Altogether this yields

$$\binom{4}{1}\binom{13}{2}\binom{13}{1}\binom{13}{1}\binom{13}{1} = 685464.$$

2. Let $X = \{x_1, x_2, \ldots, x_n\}$.

   (a) How many binary relations on X are there?

   (b) How many binary relations on X are reflexive?

   (c) How many binary relations on X are reflexive and symmetric?

   (d) Explain why there are B_n binary relations on X that are reflexive, symmetric, and transitive.

(a) A binary relation R on X is a subset $R \subseteq X \times X$. There are $|\mathcal{P}(X \times X)| = 2^{|X \times X|} = 2^{n^2}$ of these subsets.

(b) Let $D = \{(x_1, x_1), \ldots, (x_n, x_n)\}$ be the diagonal of $X \times X$. A relation $R \subseteq X \times X$ is reflexive if and only if it contains D. Therefore, a binary relation R on X is reflexive exactly when it has the form $R = D \cup S$ for some subset $S \subseteq ((X \times X) - D)$. (This says that R must contain the full diagonal D and some part S of the off-diagonal, $(X \times X) - D$.) To count such relations, we need to count the number of choices for S, which is $|\mathcal{P}((X \times X) - D)| = 2^{|((X \times X) - D)|} = 2^{n^2 - n}$.

(c) To count the reflexive, symmetric relations we must count the relations of the form $R = D \cup S$ for some symmetric subset $S \subseteq ((X \times X) - D)$. Let $U = \{(x_i, x_j) \in X \times X \mid i < j\}$ be the "upper half" of the off-diagonal, and let $L = \{(x_i, x_j) \in X \times X \mid i > j\}$ be the "lower half". A typical reflexive, symmetric relation has the form $R = D \cup U_0 \cup L_0$ where $U_0 \subseteq U$, $L_0 \subseteq L$, and $(x_i, x_j) \in U_0 \Leftrightarrow (x_j, x_i) \in L_0$. To count these, it is enough to count the possibilities for U_0 (which is the part of R in the upper half of $X \times X$). This number is $|\mathcal{P}(U)| = 2^{|U|} = 2^{\binom{n}{2}} = 2^{(n^2-n)/2}$.

(d) Any reflexive, symmetric, transitive relation on X is an equivalence relation on X. There is a bijective correspondence between the set of equivalence relations on X and the set of partitions of X. The number B_n counts the number of partitions of an n-element set, so must also count the number of equivalence relations on any n-element set (like X).

3. These problems are about seating people at a round table. Two seating arrangements are considered the same if they differ by a rotation. (So, for example, the arrangement ABCDEF is the same as BCDEFA.)

   (a) How many ways are there to seat 3 couples at a round table?

   (b) What if couples must sit together?

   (c) What if couples are not allowed to sit together?

(a) Order the 6 people in a sequence. (There are 6! ways to do this.) Now choose and fix one chair, and call it the "head chair". Seat the sequence of people around the table by having the first person sit in the head chair, then second person sit to the left of the first person, the third next to the left, and so on. Call this seating, where we specify which is the head chair, a "sequential seating".

Now define an equivalence relation on sequential seatings. Call two seatings "equivalent" if they differ by a rotation. That is, any seating ABCDEF is equivalent to BCDEFA and CDEFAB and DEFABC and EFABCD and FABCDE. This defines an equivalence relation on the set of sequential seatings where each equivalence class has 6 elements. The number of equivalence classes, which is $6!/6 = 120$ counts the number of "circular seatings".

(b) Now we assume that couples must sit together. Assume that the couples are $\{A, \alpha\}$, $\{B, \beta\}$, $\{C, \gamma\}$. Consider these 2-element sets to be "blocks" which we wish to arrange around the table in a circular manner. There are three blocks to arrange around the table, so by an argument like that in Part (a) there are $3!/3 = 2$ ways to arrange these blocks.

Next, within the block $\{A, \alpha\}$ we have to choose whether to seat the couple as $A\alpha$ or as αA. There are 2 choices for each block, so the total number of arrangements is

$$\begin{aligned}&(\text{Arrange the 3 blocks})(\text{order couple } A\alpha)(\text{order couple } B\beta)(\text{order couple } C\gamma)\\ &= (2)(2)(2)(2) = 16.\end{aligned}$$

This is the number of ways to seat the three couples around the table if the couples must sit together.

(c) We use inclusion/exclusion to count the number of arrangements where the couples are not allowed to sit together. That is, let S be the number of all circular seatings of the couples $\{A, \alpha\}$, $\{B, \beta\}$, and $\{C, \gamma\}$, let $S_{A\alpha}$ be the number of seatings where A and α sit together, let $S_{B\beta}$ be the number of seatings where B and β sit together, and let $S_{C\gamma}$ be the number of seatings where C and γ sit together. Our goal is to calculate $|S| - |S_{A\alpha} \cup S_{B\beta} \cup S_{C\gamma}|$.

- $|S| = 6!/6 = 120$.
- $|S_{A\alpha}| = (5!/5) \cdot (2) = 48$. (Circularly order 5 blocks $\{A, \alpha\}, \{B\}, \{\beta\}, \{C\}, \{\gamma\}$, then choose one of the two arrangements of the first block as either $A\alpha$ or αA.)
- $|S_{A\alpha} \cap S_{B,\beta}| = (4!/4)(2)(2) = 24$. (Circularly order 4 blocks $\{A, \alpha\}, \{B, \beta\}, \{C\}$, $\{\gamma\}$, then arrange the order of elements in the first two blocks.)
- $|S_{A\alpha} \cap S_{B,\beta} \cap S_{C,\gamma}| = (3!/3)(2)(2)(2) = 16$.

Hence, there are

$$|S| - |S_{A\alpha} \cup S_{B\beta} \cup S_{C\gamma}| = 120 - \binom{3}{1} \cdot 48 + \binom{3}{2} \cdot 24 - \binom{3}{3} \cdot 16 = 32$$

ways to arrange the six people if couples are not allowed to sit together.