Counting Problems

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In general, the Sum Rule is suggested when (exclusive) "OR" is being counted, while the Product Rule is suggested when (independent) "AND" is being counted.

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Here we used the fact that a subset $S \subseteq X$ can be described by specifying whether $1 \in S$ AND specifying whether $2 \in X$, etc.

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