

Cardinal and ordinal numbers.

Cardinal numbers (one, two three) are used to measure quantity, while ordinal numbers (first, second, third) are used put things in order.

Definition 1. (Ordinals)

- (1) A set T is *transitive* if $R \in S \in T$ implies $R \in T$.
- (2) An *ordinal (number)* is a transitive set of transitive sets.

The smallest ordinals are

$$\begin{aligned} 0 &:= \emptyset \\ 1 &:= \{0\} \\ 2 &:= \{0, 1\} \\ &\vdots \\ \omega &:= \{0, 1, 2, \dots\} \\ \omega + 1 &:= \{0, 1, 2, \dots, \omega\} \end{aligned}$$

We order ordinals by $\alpha < \beta \iff \alpha \in \beta$. Some basic properties of ordinals are

- (1) (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold.
- (2) Every ordinal is the set of its predecessors.
- (3) There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)
- (4) (Well Ordering Theorem, Zermelo) Every set can be enumerated by an ordinal. (That is, for every set X there is an ordinal α and a bijection $f : \alpha \rightarrow X$.)

The Well Ordering Theorem allows us to count any set, but the ordinal α that appears in it is not unique. For example, it is clear that the identity is a bijection $f : \omega \rightarrow \omega$, but we saw in class that there is a bijection $g : \omega + 1 \rightarrow \omega$.

This non-uniqueness implies that the ordinal numbers are not appropriate for measuring size. For this we introduce cardinal numbers.

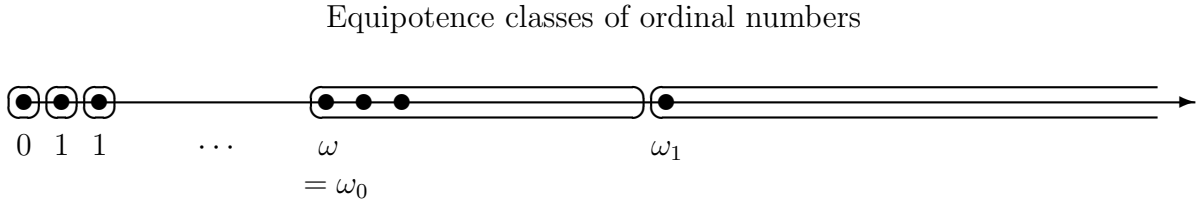
Definition 2. (Equipotence, Finiteness, Countability)

- (1) $|A| = |B|$ means there is a bijection $f : A \rightarrow B$. We read this “The cardinality of A is equal to the cardinality of B ”. When $|A| = |B|$ we say that A and B are *equipotent*.
- (2) $|A| \leq |B|$ means there is an injection $g : A \rightarrow B$.
- (3) $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
- (4) X is *finite* if it is equipotent with a natural number.
- (5) X is *infinite* if it is not finite.
- (6) X is *countably infinite* if it is equipotent with ω .
- (7) X is *countable* if it is finite or countably infinite.
- (8) X is *uncountable* if it is not countable.

Theorem 3. (Cantor-Bernstein-Schröder) If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Corollary 4. $|\mathcal{P}(\mathbb{N})| = |(0, 1)| = |\mathbb{R}|$

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.



The key features of this figure are

- (1) Equipotence classes are intervals. The classes of natural numbers are singletons.
- (2) Every equipotence class has a least element. (Such elements are called *initial ordinals*.)
- (3) For every equipotence class, there is a strictly larger class.

To measure size, we pick one ordinal from each equipotence class. Since each class has a least element, that one is the natural choice.

Definition 5. A **cardinal number** is an initial ordinal.

When discussing cardinals, it is common to use the symbols $\aleph_0, \aleph_1, \aleph_2$ in place of $\omega_0, \omega_1, \omega_2$, ETC. \aleph (aleph) is the first letter of the Hebrew alphabet. We read \aleph_0 as “aleph zero” or “aleph naught”. The first few cardinals are $0, 1, 2, \dots, \aleph_0, \aleph_1, \dots$

If κ is a cardinal number, then we might write $|X| = \kappa$ to mean $|X| = |\kappa|$, i.e., there is a bijection $f : \kappa \rightarrow X$. We do this even for finite cardinals, so $|X| = k$ for $k \in \mathbb{N}$ means there is a bijection $f : k \rightarrow X$.

We can refine the Well Ordering Theorem to say:

Theorem 6. *Every set can be enumerated by a unique cardinal number. (For every set X , there is a unique cardinal κ for which there is a bijection $f : \kappa \rightarrow X$.)*

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Theorem 7. (*Cantor’s Theorem*) *If X is a set, then $|X| < |\mathcal{P}(X)|$.*