What is a function?

First-level answer.

A *function from A to B* is a relation from A to B that satisfies the function rule.

A person who already knows what a relation is and what the function rule is will understand through this definition what a function is. Both *relation* and *function rule* are etymologically more primitive than *function*.

More fully unravelled answer.

(1) (function from A to B)

f is a function from A to B if f is a relation from A to B that satisfies the function rule.

(a) (relation from A to B)

A relation from A to B is a *subset* of the *Cartesian product* $A \times B$.

(i) (subset)

X is a subset of Y if $z \in X$ implies $z \in Y$.

(ii) (Cartesian product $A \times B$)

The Cartesian product $A \times B$ is the set

$$\{x \in \mathcal{PP}(A \cup B) \mid x = (a, b), a \in A, b \in B\}.$$

Here (a, b) is the ordered pair with 1st coordinate a and 2nd coordinate b. If A and B are sets, then $A \times B$ can be shown to be a set using the Axioms of Separation, Power Set, and Union.

(A) (ordered pair)

The ordered pair (a, b) is the set $\{\{a\}, \{a, b\}\}$. If a and b are sets, then (a, b) can be shown to be a set using the Pairing Axiom three times.

(B) (1st coordinate of an ordered pair)

If $(a, b) = \{\{a\}, \{a, b\}\}$, then the first coordinate of (a, b) is a. (A theorem was proved to show that this makes sense.)

(C) (2nd coordinate of an ordered pair)

If $(a, b) = \{\{a\}, \{a, b\}\}$, then the second coordinate of (a, b) is b.

(b) (function rule)

A relation R from A to B satisfies the function rule if for every $a \in A$ there exists exactly one $b \in B$ such that the ordered pair (a, b) is an element of R.

(i) (ordered pair)

See (1)(a)(ii)(A) above.