Torsion abelian groups

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$f_A(\sigma) = 0$ if σ is an infinite ordinal, since no nonzero element of A is infinitely p-divisible.

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The torsion subgroup B of $\prod_{k=1}^{\infty} \mathbb{Z}_{p^k}$ has all of these properties, but $B \not\cong A$ since B is uncountable while A is countable. If B were a direct sum of cyclic p-groups, then we would have to have $B \cong A$, so this B is an example of a reduced abelian p-group that is not a direct sum of cyclic groups.

The bounded case

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Prüfer's First Theorem. (Theorem 4.3.5) An abelian group satisfying the identity $nx = 0$, $n \neq 0$, is a direct sum of cyclic groups satisfying $nx = 0$.

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Proof goes here.

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Example. If A be the torsion subgroup of $\prod_{k=1}^{\infty} \mathbb{Z}_{p^k}$, then $C = \oplus_{k=1}^{\infty} \mathbb{Z}_{p^k}$ and $D = A/C$ work. The sequence does not split.