Semisimple k-algebras

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- The category of $M_n(k)$-modules has a unique simple member $S = k^n$, and every $M_n(k)$-module is isomorphic to $\oplus^\kappa S$ for uniquely determined κ.
- (Restriction of scalars) If $\varphi\colon R \to T$ is an algebra/ring hom, then any T-module M may be viewed as an R-module: $r \in R$, $m \in M$, $r \cdot m := \varphi(r) \cdot m$.
- If M is an $R \times T$-module, then $M \cong P \oplus Q$ where P is an R-module and Q is a T-module. A hom $\varphi\colon M \to M'$ corresponds to a product hom $\varphi_R \times \varphi_T$ where $\varphi_R\colon P \to P'$ and $\varphi_T\colon Q \to Q'$.
- The category of $\prod_{j=1}^k M_{n_j}(\mathbb{C})$-modules has finitely many simple members, every module is a direct sum of simple modules, and the number of summands isomorphic to given simple module is uniquely determined.

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- If $\rho\colon G \to \mathrm{Sym}(X)$ is a permutation representation of G on the set X, then there is a corresponding linear representation $\widehat{\rho}\colon G \to \mathrm{End}_k(V)$ for $V = \oplus_{x\in X} kx$ defined by $\widehat{\rho}(g)(\sum \alpha_x \cdot x) = \sum \alpha_x \rho(g)(x)$. $\widehat{\rho}$ is also called a “permutation representation”.