Representations and their characters

$$\rho_{V_i} : G \to \mathrm{GL}(V_i)$$

G	1 1	k_2 g_2	$\cdots$ $\cdots$	k_r g_r
χ_1	1	1	$\cdots$	1
χ_2	d_2	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	d_r	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

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S_3	1 1	3 (1 2)	2 (1 2 3)
χ	3	1	0

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χ_{reg}	$\lvert G\rvert$	0	$\cdots$	0

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The isomorphism $\mathbb{C}[G] \cong M_{n_1}(\mathbb{C}) \times \cdots \times M_{n_r}(\mathbb{C})$ establishes that the regular module is a direct sum of simple summands: n_i isomorphic summands of dimension n_i for each $i = 1, \dots, r$.

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(In particular, $\chi_{\mathrm{reg}} = \sum_{i=1}^{r} n_i \cdot \chi_i$.)

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The observation on this page reduces the problem of determining the characters of permutation representations to the subcase of transitive actions.

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Hence $\chi_{U^*}(g) = \overline{\chi_U(g)}$.

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Hence $\chi_{\mathrm{Hom}_{\mathbb{C}}(U,V)} = \chi_{V \otimes U^*} = \chi_V \chi_{U^*} = \overline{\chi_U} \chi_V$.

Fixed-point submodule

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In particular, this shows that, for a given character, the average of its values over G is an integer.