Orthogonality

G	1 1	k_2 g_2	$\cdots$ $\cdots$	k_r g_r
χ_1	1	1	$\cdots$	1
χ_2	d_2	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	d_r	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

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Thm.

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Cor. If $\chi_i, \chi_j \in \mathrm{Irr}(G)$, then $\langle \chi_i, \chi_j \rangle = \dim_{\mathbb{C}}(\mathrm{Hom}_{\mathbb{C}[G]}(S_i, S_j)) = \delta_{ij}$.

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Thm. If $g, h \in G$ are not conjugate, then $\sum_{\chi\in\mathrm{Irr}(G)} \overline{\chi(g)}\chi(h) = 0$.
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χ_2	1	1	ω	ω^2
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χ_4	3	-1	0	0