Inner Product

G	1 / 1	k_2 / g_2	$\cdots$ / $\cdots$	k_r / g_r
χ_1	1	1	$\cdots$	1
χ_2	d_2	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	d_r	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

Inner product

Inner product

Let $\alpha, \beta \colon G \to \mathbb{C}$ be functions.

Inner product

Let $\alpha, \beta \colon G \to \mathbb{C}$ be functions.

Write $\langle \alpha, \beta \rangle$ for the quantity $\frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g)$

Inner product

Let $\alpha, \beta \colon G \to \mathbb{C}$ be functions.

Write $\langle \alpha, \beta \rangle$ for the quantity $\frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g) = \int_G \overline{\alpha} \beta \, dg$,

Inner product

Let $\alpha, \beta \colon G \to \mathbb{C}$ be functions.

Write $\langle \alpha, \beta \rangle$ for the quantity $\frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g) = \int_G \overline{\alpha} \beta \, dg$, which is a hermitian inner product on $\mathbb{C}^G$.

Inner product

Let $\alpha, \beta \colon G \to \mathbb{C}$ be functions.

Write $\langle \alpha, \beta \rangle$ for the quantity $\frac{1}{|G|} \sum_{g \in G} \overline{\alpha(g)} \beta(g) = \int_G \overline{\alpha} \beta \, dg$, which is a hermitian inner product on $\mathbb{C}^G$.

(It is linear in its second variable and antilinear in its first variable.)

Example

Example. The inner product of irreducible characters of S_3

S_3	1 1	3 (1 2)	2 (1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Example

Example. The inner product of irreducible characters of S_3

S_3	1 1	3 (1 2)	2 (1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Unmerge the columns:

S_3	1	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	-1	1	1
χ_3	2	0	0	0	-1	-1

Example

Example. The inner product of irreducible characters of S_3

S_3	1 1	3 (1 2)	2 (1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Unmerge the columns:

S_3	1	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	-1	1	1
χ_3	2	0	0	0	-1	-1

$$\langle \chi_2, \chi_3 \rangle = \frac{1}{6}(\overline{(1)}(2) + \overline{(-1)}(0) + \overline{(-1)}(0) + \overline{(-1)}(0) + \overline{(1)}(-1) + \overline{(1)}(-1))$$

Example

Example. The inner product of irreducible characters of S_3

S_3	1 1	3 $(1\ 2)$	2 $(1\ 2\ 3)$
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Unmerge the columns:

S_3	1	$(1\ 2)$	$(1\ 3)$	$(2\ 3)$	$(1\ 2\ 3)$	$(1\ 3\ 2)$
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	-1	1	1
χ_3	2	0	0	0	-1	-1

$$\langle \chi_2, \chi_3 \rangle = \frac{1}{6}(\overline{(1)}(2) + \overline{(-1)}(0) + \overline{(-1)}(0) + \overline{(-1)}(0) + \overline{(1)}(-1) + \overline{(1)}(-1)) = 0$$

Example. The inner product of irreducible characters of S_3

S_3	1 1	3 (1 2)	2 (1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

Unmerge the columns:

S_3	1	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	-1	1	1
χ_3	2	0	0	0	-1	-1

$$\begin{aligned} \langle \chi_2, \chi_3 \rangle &= \tfrac{1}{6}(\overline{(1)}(2) + \overline{(-1)}(0) + \overline{(-1)}(0) + \overline{(-1)}(0) + \overline{(1)}(-1) + \overline{(1)}(-1)) = 0 \\ \langle \chi_3, \chi_3 \rangle &= \tfrac{1}{6}(2^2 + 0^2 + 0^2 + 0^2 + (-1)^2 + (-1)^2) = 1 \end{aligned}$$

Example

Example

We don't have to unmerge the columns to evaluate the inner product of characters if we take the conjugacy class sizes into account.

Example

We don't have to unmerge the columns to evaluate the inner product of characters if we take the conjugacy class sizes into account.

G	1 1	k_2 g_2	$\cdots$ $\cdots$	k_r g_r
χ_1	1	1	$\cdots$	1
χ_2	d_2	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	d_r	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

Example

We don't have to unmerge the columns to evaluate the inner product of characters if we take the conjugacy class sizes into account.

G	1 1	k_2 g_2	$\cdots$ $\cdots$	k_r g_r
χ_1	1	1	$\cdots$	1
χ_2	d_2	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	d_r	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

$$\begin{aligned}\langle \chi_i, \chi_j \rangle &= \frac{1}{|G|}(\overline{(d_i)}(d_j) + \underline{k_2}(\overline{(\chi_i(g_2))}(\chi_j(g_2)) + \cdots + \underline{k_r}(\overline{(\chi_i(g_r))}(\chi_j(g_r))) \\ &= \frac{1}{|G|}\sum_{\ell=1}^{r} k_\ell \overline{(\chi_i(g_\ell))}(\chi_j(g_\ell)\end{aligned}$$

Most important facts

Most important facts

- If U and V are f.d. $\mathbb{C}[G]$-modules, then $\langle \chi_U, \chi_V \rangle$ equals the dimension of the $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathbb{C}[G]}(U, V)$.

Most important facts

- If U and V are f.d. $\mathbb{C}[G]$-modules, then $\langle \chi_U, \chi_V \rangle$ equals the dimension of the $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathbb{C}[G]}(U, V)$.

Most important facts

- If U and V are f.d. $\mathbb{C}[G]$-modules, then $\langle \chi_U, \chi_V \rangle$ equals the dimension of the $\mathbb{C}$-vector space $\mathrm{Hom}_{\mathbb{C}[G]}(U, V)$.
- The irreducible characters form an orthonormal basis for the space of class functions under the inner product $\langle \_, \_ \rangle$.