What everyone already knows about groups.

(Dummit & Foote Chapters 1-9, approximately.)

(1) Definitions. Language. Terminology, like 'order of a group or element', 'exponent of a group', 'index of a subgroup'. Notation, like $|G|$, G/N, $[G : H]$, $[x, y] = x^{-1}y^{-1}xy$, $x^y = y^{-1}xy$. Easy syntactic results, like: the group axioms imply (i) $(x^{-1})^{-1} = x$ and (ii) $(xy)^{-1} = y^{-1}x^{-1}$.

(2) Examples:
- (a) $\mathbb{Z}$, $\mathbb{Z}_n$, S_n, A_n, D_n, D_∞, Q_8, $\mathrm{GL}_n(\mathbb{F})$, $\mathrm{SL}_n(\mathbb{F})$, $\mathrm{PSL}_n(\mathbb{F})$, $\mathrm{O}_n(\mathbb{F})$.
  - (i) Know how to compute in these groups, e.g. know modular arithmetic, cycle decomposition of permutations, matrix arithmetic.
- (b) Groups of small order: know a complete list of groups of order < 16.

(3) H, S, P:
- (a) Homomorphisms. Inner/outer automorphisms.
- (b) Subgroups/normal subgroups. Subgroups of index 2 are normal.
- (c) Quotient groups.
- (d) Exact sequences.
- (e) Lattices of subgroups/normal subgroups.
- (f) Simple groups. Simplicity of A_n.
- (g) Left invariant equivalence relations on groups. Cosets. Double cosets.
- (h) Characterization of products and of semidirect products.

(4) Basic structure theorems:
- (a) Cayley Representation Theorem.
- (b) Noether's Isomorphism Theorems. (3)
- (c) Correspondence Theorem.
- (d) Jordan-Hölder Theorem.
- (e) Lagrange's Theorem.
- (f) Cauchy's Theorem.
- (g) Sylow's Theorems.

(5) Commutator theory:
- (a) Abelianness.
- (b) Centralizers.
- (c) Nilpotence: center, ascending/descending central series. Frattini argument.
- (d) solvability: derived group/series.

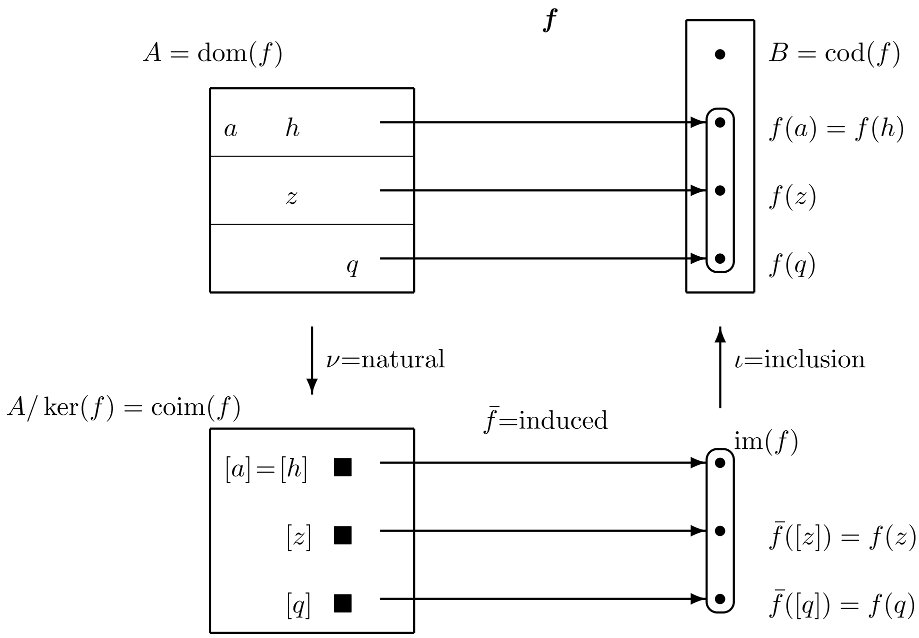
(6) Group actions:
- (a) Equivalent definitions, terminology (G-set, orbit, (k-)transitive action, (semi)regular action, homomorphisms/automorphisms).
- (b) Orbit-Stabilizer Theorem.
- (c) Class equation.
- (d) Structure Theorem for G-sets.
- (e) Conjugacy action on elements, subgroups. Conjugacy classes, normalizers. Conjugacy classes in S_n, A_n, $\mathrm{GL}_n(\mathbb{F})$.

(7) Abelian groups:
- (a) Structure Theorem for finitely generated abelian groups.
- (b) Invariant factor form versus elementary divisor form.

(8) Infinite groups:
- (a) Free groups, reduced words.
- (b) Presentations. Universal property of presentations.
- (c) Coproducts of groups given by presentations.

Terminology for functions.

Let A and B be sets and let $f\colon A \to B$ be a function from A to B. There are sets and functions related to A, B and f that have special names.



(1) The *image* of f is $\mathrm{im}(f) = f[A] = \{b \in B \ : \ \exists a \in A(f(a) = b)\}$. The image of a subset $U \subseteq A$ is $f[U] = \{b \in B \ : \ \exists u \in U(f(u) = b)\}$.
(2) The *preimage* or *inverse image* of a subset $V \subseteq B$ is $f^{-1}[V] = \{a \in A \ : \ f(a) \in V\}$.
(3) The preimage of a singleton $\{b\}$ is written $f^{-1}(b)$ and sometimes called the *fiber* of f *over* b. The fiber $f^{-1}(f(a))$ containing the element a is sometimes written $[a]$.
(4) The *kernel* of f is $\ker(f) = \{(a, a') \in A^2 \ : \ f(a) = f(a')\}$, which is an equivalence relation on A.
(5) The *kernel class* of an element $a \in A$ is the same as the fiber $[a]$ that contains it. It is often written $a/\ker(f)$ to emphasize that it is a kernel class.
(6) The *coimage* of f is the set

$$\mathrm{coim}(f) = \{f^{-1}(b) \ : \ b \in \mathrm{im}(f)\} = \{[a] \ : \ a \in A\} = A/\ker(f)$$

of all nonempty fibers.
(7) The *natural map* is $\nu\colon A \to \mathrm{coim}(f)\colon a \mapsto [a]$. (More generally, for any equivalence relation E on A, the natural map is $\nu\colon A \to A/E\colon a \mapsto a/E$.)
(8) The *inclusion map* is $\iota\colon \mathrm{im}(f) \to B\colon b \mapsto b$. (More generally, for any subset S of B, the inclusion map is $\iota\colon A \to B\colon b \mapsto b$.)
(9) The *induced map* is $\bar{f}\colon \mathrm{coim}(f) \to \mathrm{im}(f)\colon [a] \mapsto f(a)$.

Some facts:

(1) The natural map is *surjective*.
(2) The inclusion map is *injective*.
(3) The induced map is *bijective*.
(4) $f = \iota \circ \bar{f} \circ \nu$. (This is the *canonical factorization* of f.)