Complex representations of finite groups

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Reducing reps of groups to reps of algebras

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Thm. There is a (natural) bijection

$$\mathrm{Hom}_{k\text{-alg}}(k[G], \mathbb{A}) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{Grp}}(G, \mathbb{A}^\times) \colon \varphi \mapsto \varphi|_G.$$

Proof.

- $\varphi(1_{k\text{-alg}}) = \varphi(1_G) = 1_{\mathbb{A}}$.
- $\varphi(G) \subseteq \mathbb{A}^\times$.
- $\varphi|_G$ a group hom.
- If $\varphi|_G = \psi|_G$, then $\varphi = \psi$.
- If $\theta \colon G \to \mathbb{A}^\times$ $(\subseteq \mathbb{A})$ is any function, there is a unique linear extension $\widehat{\theta} \colon k[G] \to \mathbb{A}$.
- $\widehat{\theta}$ is a k-algebra homomorphism.
- $\widehat{\theta}|_G = \theta$.
- $\square$

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(Simplicity and semisimplicity are independent properties for rings.)

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8. $\mathbb{A}$ is isomorphic to a finite direct product of matrix algebras over division algebras.

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