Group Theory

Homework Assignment III

Read all of the statements of the theorems in Sections 8.1-8.3 and 8.5.1-8.5.2, and the proofs of the interesting theorems.

CHAPTER	**PROBLEMS**
8.1	**1 (I)**
8.2	**5 (II)**
8.3	**11 (III)**
8.5	**1 (IV)**

ADDITIONAL PROBLEMS

1. (I) Show that $\mathrm{Spec}_{\mathbf{Grp}}(k) = 2$ iff one the following is true.

(a) $k = p_1 \cdots p_r$ is square-free and there is exactly one relation $p_i \mid (p_j - 1)$ among the prime divisors.

(b) The prime factorization of k is $p_1 \cdots p_r \cdot q^2$ (exactly one exponent $\neq 1$, and that exponent is 2), and there are no relations among the primes. Here p_i or q is related to p_j means "$p_i \mid (p_j - 1)$" or "$q \mid (p_j - 1)$", while p_j is related to q^2 means "$p_j \mid (q^2 - 1)$".

2. (II) Give two proofs of the following claim, one using character theory and one not using character theory.

Claim. If $\omega_1, \ldots, \omega_p$ are p-th roots of unity, and $\omega_1 + \cdots + \omega_p = 0$, then these roots of unity are distinct.

[Hint for the character-theoretic proof: Define $\rho\colon \mathbb{Z}_p \to \mathbf{GL}_p(\mathbb{C})$ by

$$1 \mapsto \begin{bmatrix} \omega_1 & 0 & \cdots & 0 \\ 0 & \omega_2 & \cdots & 0 \\ \vdots & & & \vdots \\ 0 & 0 & \cdots & \omega_p \end{bmatrix}.$$

Show that the character afforded by ρ is the regular character.]

3. (III) Show that if $\chi \in \mathbf{Irr}(G)$ and $\chi(1) > 1$, then $\chi(g) = 0$ for some $g \in G$. [Hints:

(a) Use Row Orthogonality to deduce that $1 = \frac{1}{|G|}\sum_{g\in G} |\chi(g)|^2$.

(b) Use the arithmetic-geometric mean inequality to show $\prod_{g\in G} |\chi(g)|^2 < 1$.
(c) Employ a norm argument to show that the norm, ν, of $\prod_{g\in G} |\chi(g)|^2$ is an integer satisfying $0 \le \nu < 1$.

4. (IV) Let G be a finite group with $g \notin G'$.
(a) Show that conjugacy classes outside of G' are contained in cosets: $g^G \subseteq gG'$ for $g \notin G'$.
(b) Show that if conjugacy classes outside of G' are equal to cosets, $\forall g \notin G'(g^G = gG')$, then every $\chi \in \mathbf{Irr}(G)$ with $\chi(1) > 1$ vanishes off of G'.

[Hint for (b): Show that the inner product of the g-th column of the character table with itself is the same whether one computes in G or in G/G', namely it is $|C_G(g)| = [G : G']$.]

5. (I) Let G be a finite group. Suppose that every $\chi \in \mathbf{Irr}(G)$ with $\chi(1) > 1$ vanishes off of G'. Show that each nonidentity coset of G' is a conjugacy class.

[Hint: Show that if $g \notin G'$ and $gh \in gG'$, then $\chi(g) = \chi(gh)$ for every $\chi \in \mathbf{Irr}(G)$. Show that this is true for the linear characters by inflation, and for the nonlinear characters by hypothesis. Conclude that $gh \in g^G$.]

Remark: Problem 5 is the converse of 4(b). That is, a group G has the property that its nonlinear characters vanish off of G' iff cosets gG' with $g \notin G'$ are conjugacy classes. Groups with these equivalent properties are called Camina groups.

6. (II) Let p be an odd prime. The two nonabelian groups of order p^3 have presentations

$$G_1 = \langle a, b \mid a^p = b^p = 1, [[a,b],b] = [[a,b],a] = 1\rangle$$

and

$$G_2 = \langle a, b \mid a^{p^2} = b^p = 1, [a,b] = a^p\rangle.$$

Let σ be a primitive p^2-th root of unity and let $\omega = \sigma^p$ be a primitive p-th root of unity. Consider the $p \times p$ matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{p-1} \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}.$$

(a) Show that $a \mapsto A$ and $b \mapsto B$ is an irreducible representation of G_1, and that $a \mapsto \sigma A$ and $b \mapsto B$ is an irreducible representation of G_2.
(b) Show that G_1 and G_2 have the same character tables.
(c) Show that the two groups can be distinguished by their determinant maps.

7. (III) Prove that a finite nonabelian simple group has no irrep of degree 2.

[Hints: Assume χ is an irrep of degree 2 afforded by some $\rho\colon G \to \mathbf{GL}_2(\mathbb{C})$. Show that G must contain an involution g, consider the possible eigenvalues of $\rho(g)$, and derive a contradiction.]

Collaboration Groups.

(I) DeLand, Kuo, Watson
(II) Gensler, Jamesson, Willson
(III) Doumont, Lyness, Shearer
(IV) Orvis, Wilson