The Group Commutator

The commutator of normal subgroups M and N is generated by all elements of the form $[m,n] := m^{-1}n^{-1}mn$, so it is natural to study this kind of "multiplication" of elements, and to consider the subgroup generated by these elements even in the case where M and N are not necessarily normal. Here are some of the basic properties of commutators of elements, subgroups and normal subgroups.

Commutator of elements. Let x, y, z be elements of some group. Define $[x_1, x_2, \ldots, x_n] = [[x_1, x_2, \ldots, x_{n-1}], x_n]$.

(i) $[x,y]^{-1} = [y,x]$.

(ii) $[xy,z] = [x,z]^y[y,z]$ and $[x,yz] = [x,z][x,y]^z$.

(iii) $[x,y^{-1}] = \left([x,y]^{y^{-1}}\right)^{-1}$ and $[x^{-1},y] = \left([x,y]^{x^{-1}}\right)^{-1}$.

(iv) (Hall-Witt Identity) $[x,y^{-1},z]^y[y,z^{-1},x]^z[z,x^{-1},y]^x = 1$.

Commutator of subgroups. Let H, K and L be subgroups.

(v) (Three subgroups lemma.) If $[H,K,L] = \{1\}$ and $[K,L,H] = \{1\}$, then $[L,H,K] = \{1\}$.

Commutator of normal subgroups. In this part, all subgroups are normal.

Order-theoretic properties of the commutator.

(vi) (Monotonicity) If $M_1 \leq M_2$ and $N_1 \leq N_2$, then $[M_1,N_1] \leq [M_2,N_2]$.

(vii) (Sub-meet) $[M,N] \leq M \cap N$.

(viii) (Commutativity) $[M,N] = [N,M]$.

(ix) (Complete additivity) $[M, \bigvee_{i\in I} N_i] = \bigvee_{i\in I}[M,N_i]$.

The last of these properties implies that for any $L, M \lhd G$ there is a largest $N \lhd G$ such that $[M,N] \leq L$. This largest N is denoted $(L : M)$. (An important special case is $C_G(M) = (\{1\} : M)$.)

HSP properties of the commutator.

(x) $[L/N, M/N] = [L,M]/N$ in G/N.

(xi) If $H \leq G$, then $[M|_H, N|_H] \leq [M,N]|_H$. (For a subgroup $L \leq G$, its restriction to H is $L|_H := L \cap H$.)

(xii) $[K_1 \times M_2, L_1 \times N_2] = [K,L]_1 \times [M,N]_2$ in the group $G_1 \times G_2$.

Solvable Groups.

Let $G^{(0)} := G$, $G^{(1)} = G' := [G,G]$, and $G^{(n+1)} := [G^{(n)}, G^{(n)}]$. Then $G \geq G^{(1)} \geq \cdots$ is the *derived series*. Each group in this series is verbal. G is *k-step solvable* if $G^{(k)} = \{1\}$ for some finite n.

(xiii) The relation "$M \sim N \Leftrightarrow [M \cap N, MN]$ is solvable" is a congruence on $\mathrm{Norm}(G)$.

Nilpotent Groups.

Let $\gamma_1(G) := G$ and $\gamma_{n+1}(G) = [G, \gamma_n(G)]$. Then $G = \gamma_1(G) \geq \cdots$ is the *descending* (or *lower*) *central series*. Each group in this series is verbal. G is *k-step nilpotent* if $\gamma_k(G) = \{1\}$ for some finite n.

Let $\zeta_0(G) := \{1\}$, $\zeta_1(G) = \zeta(G) := (\{1\} : G)$, $\zeta_{\kappa+1}(G) := (\zeta_\kappa(G) : G)$, and $\zeta_\lambda(G) = \bigcup_{\kappa<\lambda} \zeta_\kappa(G)$ when λ is a limit ordinal. Then $\{1\} \leq \zeta_1(G) \leq \zeta_2(G) \leq \cdots$ is the *ascending* (or *upper*) *central series*. The groups in this series are characteristic. The group $\zeta(G)$ is the *center* of G, and the union of all $\zeta_\kappa(G)$ is the *hypercenter*.

Let $\{1\} = G_0 \leq \cdots \leq G_n = G$ be any series of normal subgroups such that $[G, G_{i+1}] \leq G_i$. Then

(xiv) $G_i \leq \zeta_i(G)$ for all i, and

(xv) $\gamma_i(G) \leq G_{n-i+1}$.

Hence $\gamma_{n+1}(G) = \{1\}$ iff $\zeta_n(G) = G$.

Examples 1.

(i) p-groups are nilpotent. (Proof #1: Use the class equation to prove that the center is nontrivial. Use induction to prove that the ascending central series cannot terminate until it reaches G.) (Proof #2: Embed the p-group P into a Sylow p-subgroup of S_n, and use the structure of such groups to prove that P has a normal subgroup of index p. Continue by induction to produce a normal series with factors isomorphic to $\mathbb{Z}_p$. Show G acts trivially on the series.)

(ii) Unipotent groups are nilpotent. (A matrix $M \in \mathrm{GL}(n, \mathbb{F})$ is *unipotent* if it satisfies $(x-1)^n = 0$, equivalently if all of its eigenvalues are 1. The group of all upper triangular unipotent matrices in $\mathrm{GL}(n, \mathbb{F})$ is callled the unipotent group, $\mathrm{U}(n, \mathbb{F})$.) (Proof of nilpotence: It helps to write unipotent matrices in the form $I + N$ where N is strictly upper triangular, and then to do calculations in the matrix ring $M_n(\mathbb{F})$. With this in mind, argue by induction that $\gamma_k(\mathrm{U}(n, \mathbb{F}))$ consists only of matrices of the form $I + N_k$ where $N_k = [a_{ij}]$ and $a_{ij} = 0$ unless $j - i \geq k$.) (Interesting fact: by considering orders, one can show that $\mathrm{U}(n, \mathbb{F}_q)$ is a Sylow p-subgroup of $\mathrm{GL}(n, \mathbb{F}_q)$ when $q = p^r$. Thus every finite p-group is embeddable in a finite unipotent p-group.)

Commutator collection.

There is a normal form for words in the class of k-step nilpotent groups. To describe it, define the *weight* of commutator word inductively by $\mathrm{wt}(x_i) = 1$ and $\mathrm{wt}([c_i, c_j]) = \mathrm{wt}(c_i) + \mathrm{wt}(c_j)$. The goal of the process is to rewrite a word $w(x_1, \ldots, x_n)$ in the form

$$x_1^{e_1} \cdots x_n^{e_n} \cdot \left(\prod_{i<j} [x_i, x_j]^{e_{ij}}\right) \cdot \left(\prod_{i<j<k} [x_i, x_j, x_k]^{e_{ijk}}\right) \cdot (\text{higher weight commutators}) \cdots .$$

The collection process is based on the strategy of moving lower weight commutators to the left by replacing yx with $xy[x,y]^{-1}$" when $\mathrm{wt}(x) < \mathrm{wt}(y)$. Nilpotence is needed to guarantee termination.

More on the three subgroup lemma.

(xvi) If $H, K, L \lhd G$, then $[H, K, L] \leq [K, L, H][L, H, K]$.

Consequences include:

(xvii) $[\gamma_i(G), \gamma_j(G)] \leq \gamma_{i+j}(G)$.
(xviii) $\gamma_i(\gamma_j(G)) \leq \gamma_{ij}(G)$.
(xix) $[\gamma_n(G), \zeta_{n+k}(G)] \leq \zeta_k(G)$.
(xx) $G^{(i)} \leq \gamma_{2^i}(G)$. (Use (xviii) repeatedly with $i = 2$.)
(This limits the derived length of a nilpotent group.)

Characterizations of nilpotence.

Theorem 2. *The following are equivalent for a finite group.*

(1) *G is nilpotent.*
(2) *Every subgroup of G is* subnormal.
(3) *If $H < G$, then $H < N_G(H)$.*
(4) *Every maximal subgroup of G is normal.*
(5) *Every Sylow subgroup of G is normal.*
(6) *G is the product of its Sylow subgroups.*