Basic Properties of Characters of Finite Groups.

In this note, G is a finite group, V is a finite dimensional $\mathbb{C}[G]$-module, $\rho_V\colon G \to \mathrm{GL}(V)$ is the corresponding representation, and $\chi_V = \mathtt{tr} \circ \rho_V$ is the character afforded by ρ_V. (The subscripts will be omitted if they are irrelevant.) $\mathrm{Irr}(G)$ is the set of all irreducible characters of G. $K_G(g)$ is the conjugacy class of g in G.

Linear algebra.

(1) $\rho(g)$ is diagonalizable.

(2) $\chi_V(1) = \mathtt{tr}(I) = \dim_{\mathbb{C}}(V)$ is the *degree* of χ_V.

(3) $\chi(hgh^{-1}) = \chi(g)$ (χ is a class function).

(4) $\chi(g)$ is the sum of $\chi(1)$ $|G|$-th roots of unity.

(5) $\chi(g^{-1}) = \overline{\chi(g)}$.

(6) $|\chi_V(g)| \leq \chi_V(1)$, with equality iff $\rho_V(g) = \omega I$ for some $|G|$-th root of unity ω.

Dimension.

(7) $|G| = \dim_{\mathbb{C}}(\mathbb{C}[G]) = \sum_{\chi\in\mathrm{Irr}(G)} \chi(1)^2$.

(8) The number of isomorphism types of simple G-modules $= \dim_{\mathbb{C}}(Z(\mathbb{C}[G]))$ = the number of conjugacy classes of G.

Kernel and center.

(9) $K_\chi := \{g \in G \mid \chi(g) = \chi(1)\}$ is a normal subgroup of G. (K_χ is the kernel of the associated representation, so it is called the *kernel* of χ.)

(10) $Z_\chi := \{g \in G \mid |\chi(g)| = \chi(1)\}$ is a normal subgroup of G containing K_χ.

(11) A subset $K \subseteq G$ is a normal subgroup iff $K = K_\chi$ for some not-necessarily-irreducible character χ.[1] Thus, $\mathrm{Irr}(G)$ determines the normal subgroups of G, the normal subgroup lattice of G, the indices $[H : K]$ between comparable normal subgroups, whether or not G is solvable, and whether or not G is nilpotent.

(12) $K_\chi \lhd Z_\chi$ and Z_χ/K_χ is cyclic.

(13) $Z_\chi/K_\chi \subseteq Z(G/K_\chi)$, with equality if $\chi \in \mathrm{Irr}(G)$.

(14) For normal subgroups $X, Y \lhd G$,

$$[G, X] \leq Y \longleftrightarrow (\forall\chi \in \mathrm{Irr}(G))(Y \leq K_\chi \to X \leq Z_\chi) \longleftrightarrow X \leq (Y : G).$$

Therefore, $\mathrm{Irr}(G)$ determines the functions $X \mapsto [G, X]$ and $Y \mapsto (Y : G)$ on $\mathrm{Norm}(G)$, and hence the descending & ascending central series.

(15) $[G, G] = \bigcap_{\chi \text{ linear}} K_\chi$.

(16) $Z(G) = \bigcap_{\chi\in\mathrm{Irr}(G)} Z_\chi$.

[1] Every $\cap$-irreducible normal subgroup is K_χ for some irreducible χ.

Constructions of G-modules, and their characters. The constant homomorphism $\rho_1\colon G \to \mathrm{GL}(1,\mathbb{C}) = \mathbb{C}^\times : g \mapsto 1$ defines the trivial G-module; its character χ_1 is called the *principal character*. The principal character satisfies $\chi_1(g) = 1$ for all g.

If G acts on the set X via the homomorphism $\rho\colon G \to \mathrm{Sym}(X)$, then it acts on the $\mathbb{C}$-space with basis X, $\mathbb{C}^X$, via the homomorphism $\widehat{\rho}\colon G \to \mathrm{GL}(\mathbb{C}^X)$ defined by $\widehat{\rho}(g)(\sum_{x\in X} a_x x) = \sum_{x\in X} a_x \rho(g)(x)$. Such a $\widehat{\rho}$ is called a *permutation representation*. The special case where $X = G$ and G acts on X by left multiplication is called the *regular representation*.

(17) A G-module U is isomorphic to a permutation representation of G if and only if U has a G-invariant basis.

(18) If $\rho_X\colon G \to \mathrm{Sym}(X)$ represents an action of G on X, then $\chi_X(g)$ equals the number of elements of X fixed by g.

(19) The character of the regular representation is $\chi_{\mathrm{reg}}(g) = \begin{cases} |G| & \text{if } g = 1; \\ 0 & \text{otherwise.} \end{cases}$

(20) The regular representation is faithful.

(21) $\chi_{U\oplus V} = \chi_U + \chi_V$.

(22) $\chi_{U\otimes V} = \chi_U \chi_V$.

(23) If $\rho_X\colon G \to \mathrm{Sym}(X)$ represents an action of G on X, $\rho_Y\colon G \to \mathrm{Sym}(Y)$ represents an action of G on Y, and $\rho_{X\times Y}\colon G \to \mathrm{Sym}(X\times Y)$ represents the product action $(g(x,y) = (gx, gy))$, then $\mathbb{C}^X \otimes \mathbb{C}^Y \cong \mathbb{C}^{X\times Y}$ and $\chi_{X\times Y} = \chi_X \chi_Y$.

(24) $\chi_{U^*} = \overline{\chi_U}$.

(25) $\chi_{\mathrm{Hom}_{\mathbb{C}}(U,V)} = \overline{\chi_U}\chi_V$.

(26) $\dim_{\mathbb{C}}(V^G) = \frac{1}{|G|}\sum_{g\in G} \chi_V(g)$.

Inner Product. Let $\alpha, \beta\colon G \to \mathbb{C}$ be functions. Write $\langle \alpha, \beta\rangle$ for $\frac{1}{|G|}\sum_{g\in G} \overline{\alpha(g)}\beta(g)$, which is the usual hermitian inner product on $\mathbb{C}^G$ weighted by the factor $1/|G|$.

(27) $\langle \chi_V, \chi_U\rangle = \langle \chi_V\overline{\chi_U}, \chi_1\rangle = \langle \chi_{\mathrm{Hom}_{\mathbb{C}}(U,V)}, \chi_1\rangle = \dim_{\mathbb{C}}\mathrm{Hom}_{\mathbb{C}[G]}(U,V) = \langle \chi_U, \chi_V\rangle$.

(28) (Row Orthogonality) If $\chi_i, \chi_j \in \mathrm{Irr}(G)$, then $\langle \chi_i, \chi_j\rangle = \frac{1}{|G|}\sum_{g\in G}\overline{\chi_i(g)}\chi_j(g) = \delta_{ij}$.

(29) (Column Orthogonality) If $g, h \in G$ are not conjugate, then $\sum_{\chi\in\mathrm{Irr}(G)} \overline{\chi(g)}\chi(h) = 0$. Otherwise this sum is $|C_G(g)|$ (the order of the centralizer of g). The second part of this claim generalizes (7) above.

(30) χ is irreducible iff $\langle \chi, \chi\rangle = 1$.

(31) χ_U is the character of a G-module U and $\{\chi_1, \ldots, \chi_r\}$ are the distinct irreducible characters of G, then $\chi_U = \sum \langle \chi_i, \chi_U\rangle \cdot \chi_i$. In particular, the isomorphism type of a G-module is determined by its character.

(32) (Burnside's Lm, Not-Burnside's Lm, Cauchy-Frobenius Lm) If G acts on X with permutation character χ_X, then the number of orbits of the action is $\langle \chi_X, \chi_1\rangle$.

(33) Let G act on X with permutation character χ_X. The action is 2-transitive iff $\chi_X = \chi_1 + \chi$ for some $\chi \in \mathrm{Irr}(G) - \{\chi_1\}$.

Integrality Properties.

(34) $\chi(g)$ is an algebraic integer.

(35) If $\chi_i \in \mathrm{Irr}(G)$, then $k_j \cdot \chi_i(g_j)/\chi_i(1)$ is an algebraic integer.

(36) If $\chi_i \in \mathrm{Irr}(G)$ and $\gcd(\chi_i(1), k_j) = 1$, then $\chi_i(g_j)/\chi_i(1)$ is an algebraic integer.

(37) If $\chi_i(g_j)/\chi_i(1)$ is an algebraic integer, then either $g_j \in Z_{\chi_i}$ or $\chi_i(g_j) = 0$.

(38) If $\chi_i \in \mathrm{Irr}(G)$, then $\chi_i(1)$ divides $|G|$.

Strengthenings:

(i) $\chi(1) \mid [G : Z_\chi]$.

(ii) $\chi(1)^2 \leq [G : Z_\chi]$.

(iii) $\chi(1) \leq [G : A]$ if $A \leq G$ and $[A, A] \leq K_\chi$.

(iv) $\chi(1) \mid [G : A]$ if $A \lhd G$ and $[A, A] \leq K_\chi$.

Example 1. (A character table.) The character table of a finite group G is the concatenation of the function tables of the irreducible characters of G. The first natural attempt to write down such a table for the group $S_3 (\cong D_3)$ would produce:

S_3	1	(1 2)	(1 3)	(2 3)	(1 2 3)	(1 3 2)
χ_1	1	1	1	1	1	1
χ_2	1	-1	-1	-1	1	1
χ_3	2	0	0	0	-1	-1

But since each character is a class function, columns corresponding to conjugate group elements are identical. A more compact representation of the same information results from identifying duplicate columns. To record the details of the identification, choose a list of representatives $1 = g_1, g_2, \ldots, g_r$ for the conjugacy classes. Now delete all columns except those indexed by these elements. Record the size $k_i := |K_G(g_i)|$ of each class above the class representative. In general, the result is the table on the left, while the table for S_3 is given on the right.

	1	k_2	$\cdots$	k_r
G	1	g_2	$\cdots$	g_r
χ_1	1	1	$\cdots$	1
χ_2	d_2	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	d_r	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

	1	3	2
S_3	1	(1 2)	(1 2 3)
χ_1	1	1	1
χ_2	1	-1	1
χ_3	2	0	-1

It is a convention to let the first column be that of the conjugacy class $\{1\}$, and the first row to be that of the principal character. (On the left, $d_i = \chi_i(1)$ is the degree of χ_i.)

References

[1] Alperin, J. L.; Bell, Rowen B., *Groups and representations.* Graduate Texts in Mathematics, 162. Springer-Verlag, New York, 1995.
(See Chapters 5 & 6: Sections 12-15.)

[2] Isaacs, I. Martin, *Character theory of finite groups.* Corrected reprint of the 1976 original AMS Chelsea Publishing, Providence, RI, 2006.
(See Chapters 1-4.)