Numbers beyond $\mathbb{N}$

$$0, 1, 2, \ldots, \omega, \omega + 1, \omega + 2, \ldots$$

Cardinal numbers versus ordinal numbers

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity,

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition.

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$.

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$.

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$. ($\bigcup T \subseteq T$)

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$. ($\bigcup T \subseteq T$)
2. An *ordinal (number)* is a transitive set of transitive sets.

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$. ($\bigcup T \subseteq T$)
2. An *ordinal (number)* is a transitive set of transitive sets.

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$. ($\bigcup T \subseteq T$)
2. An *ordinal (number)* is a transitive set of transitive sets.

The smallest ordinals are

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$. ($\bigcup T \subseteq T$)
2. An *ordinal (number)* is a transitive set of transitive sets.

The smallest ordinals are

$$0 \quad := \emptyset$$

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$. ($\bigcup T \subseteq T$)
2. An *ordinal (number)* is a transitive set of transitive sets.

The smallest ordinals are

$$\begin{aligned} 0 &:= \emptyset \\ 1 &:= \{0\} \end{aligned}$$

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$. ($\bigcup T \subseteq T$)
2. An *ordinal (number)* is a transitive set of transitive sets.

The smallest ordinals are

$$\begin{aligned} 0 &:= \emptyset \\ 1 &:= \{0\} \\ 2 &:= \{0, 1\} \end{aligned}$$

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$. ($\bigcup T \subseteq T$)
2. An *ordinal (number)* is a transitive set of transitive sets.

The smallest ordinals are

$$\begin{aligned} 0 &:= \emptyset \\ 1 &:= \{0\} \\ 2 &:= \{0, 1\} \\ 3 &:= \{0, 1, 2\} \end{aligned}$$

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$. ($\bigcup T \subseteq T$)
2. An *ordinal (number)* is a transitive set of transitive sets.

The smallest ordinals are

$$\begin{aligned} 0 &:= \emptyset \\ 1 &:= \{0\} \\ 2 &:= \{0, 1\} \\ 3 &:= \{0, 1, 2\} \\ &\vdots \\ \omega &:= \{0, 1, 2, \ldots\} \qquad = \mathbb{N} \end{aligned}$$

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$. ($\bigcup T \subseteq T$)
2. An *ordinal (number)* is a transitive set of transitive sets.

The smallest ordinals are

$$\begin{aligned} 0 &:= \emptyset \\ 1 &:= \{0\} \\ 2 &:= \{0, 1\} \\ 3 &:= \{0, 1, 2\} \\ &\vdots \\ \omega &:= \{0, 1, 2, \ldots\} && = \mathbb{N} \\ \omega + 1 &:= \{0, 1, 2, \ldots, \omega\} && = S(\mathbb{N}) \end{aligned}$$

Cardinal numbers versus ordinal numbers

Cardinal numbers (one, two three) are used to measure quantity, while **ordinal numbers** (first, second, third) are used put things in order.

Definition. (Ordinals)

1. A set T is *transitive* if $R \in S \in T$ implies $R \in T$. ($\bigcup T \subseteq T$)
2. An *ordinal (number)* is a transitive set of transitive sets.

The smallest ordinals are

$$\begin{aligned}
0 &:= \emptyset \\
1 &:= \{0\} \\
2 &:= \{0, 1\} \\
3 &:= \{0, 1, 2\} \\
&\vdots \\
\omega &:= \{0, 1, 2, \dots\} && = \mathbb{N} \\
\omega + 1 &:= \{0, 1, 2, \dots, \omega\} && = S(\mathbb{N}) \\
\omega + 2 &:= \{0, 1, 2, \dots, \omega, \omega + 1\} && = SS(\mathbb{N})
\end{aligned}$$

Properties of the ordinal numbers

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$.

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold.

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold.

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.
3. There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.
3. There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.
3. There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)
4. The class of ordinals is "well ordered".

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.
3. There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)
4. The class of ordinals is "well ordered".

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.
3. There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)
4. The class of ordinals is "well ordered". (This means that any nonempty set X of ordinals has a least element, namely $\bigcap X$.)

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.
3. There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)
4. The class of ordinals is "well ordered". (This means that any nonempty set X of ordinals has a least element, namely $\bigcap X$.)
5. (Well Ordering Theorem) Every set can be enumerated by an ordinal. (That is, for every set X there is an ordinal α and a bijection $f : \alpha \to X$.)

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.
3. There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)
4. The class of ordinals is "well ordered". (This means that any nonempty set X of ordinals has a least element, namely $\bigcap X$.)
5. (Well Ordering Theorem) Every set can be enumerated by an ordinal. (That is, for every set X there is an ordinal α and a bijection $f\colon \alpha \to X$.)

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.
3. There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)
4. The class of ordinals is "well ordered". (This means that any nonempty set X of ordinals has a least element, namely $\bigcap X$.)
5. (Well Ordering Theorem) Every set can be enumerated by an ordinal. (That is, for every set X there is an ordinal α and a bijection $f\colon \alpha \to X$.)

The Well Ordering Theorem allows us to count any set, but the ordinal α that appears in it is not unique.

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.
3. There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)
4. The class of ordinals is "well ordered". (This means that any nonempty set X of ordinals has a least element, namely $\bigcap X$.)
5. (Well Ordering Theorem) Every set can be enumerated by an ordinal. (That is, for every set X there is an ordinal α and a bijection $f\colon \alpha \to X$.)

The Well Ordering Theorem allows us to count any set, but the ordinal α that appears in it is not unique. For example, there are bijections $f\colon \omega \to \omega$ and $g\colon \omega + 1 \to \omega$, so ω can be counted by both ω and $\omega + 1$.

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.
3. There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)
4. The class of ordinals is "well ordered". (This means that any nonempty set X of ordinals has a least element, namely $\bigcap X$.)
5. (Well Ordering Theorem) Every set can be enumerated by an ordinal. (That is, for every set X there is an ordinal α and a bijection $f \colon \alpha \to X$.)

The Well Ordering Theorem allows us to count any set, but the ordinal α that appears in it is not unique. For example, there are bijections $f \colon \omega \to \omega$ and $g \colon \omega + 1 \to \omega$, so ω can be counted by both ω and $\omega + 1$.
This non-uniqueness implies that the ordinal numbers are not appropriate for measuring size.

Properties of the ordinal numbers

We order ordinals by $\alpha < \beta$ iff $\alpha \in \beta$. Some basic properties of ordinals are

1. (Trichotomy) If α and β are ordinals, then exactly one of $\alpha < \beta$, $\alpha = \beta$, or $\beta < \alpha$ must hold. This says that the ordinals are linearly ordered by $\in$.
2. Every ordinal is the set of its predecessors.
3. There is no infinite descending chain of ordinals. (Because of the Axiom of Foundation.)
4. The class of ordinals is "well ordered". (This means that any nonempty set X of ordinals has a least element, namely $\bigcap X$.)
5. (Well Ordering Theorem) Every set can be enumerated by an ordinal. (That is, for every set X there is an ordinal α and a bijection $f\colon \alpha \to X$.)

The Well Ordering Theorem allows us to count any set, but the ordinal α that appears in it is not unique. For example, there are bijections $f\colon \omega \to \omega$ and $g\colon \omega + 1 \to \omega$, so ω can be counted by both ω and $\omega + 1$.
This non-uniqueness implies that the ordinal numbers are not appropriate for measuring size. For this we introduce cardinal numbers.

Equipotence, Finiteness, Countability

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f : A \to B$. We read this "The cardinality of A is equal to the cardinality of B".

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f : A \to B$. We read this "The cardinality of A is equal to the cardinality of B".

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f\colon A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent*

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f : A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f: A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g: A \to B$.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f: A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g: A \to B$.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f : A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g : A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f\colon A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g\colon A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f\colon A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g\colon A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f: A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g: A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f : A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g : A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f : A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g : A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f\colon A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g\colon A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f: A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g: A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f\colon A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g\colon A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f : A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g : A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f\colon A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g\colon A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.
8. X is *uncountable* if it is not countable.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f\colon A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g\colon A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.
8. X is *uncountable* if it is not countable.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f\colon A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g\colon A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.
8. X is *uncountable* if it is not countable.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f\colon A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g\colon A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.
8. X is *uncountable* if it is not countable.

Examples.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f\colon A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g\colon A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.
8. X is *uncountable* if it is not countable.

Examples. Any $n \in \omega$ is finite.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f : A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g : A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.
8. X is *uncountable* if it is not countable.

Examples. Any $n \in \omega$ is finite. ω is countably infinite.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f: A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g: A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.
8. X is *uncountable* if it is not countable.

Examples. Any $n \in \omega$ is finite. ω is countably infinite. $\mathbb{R}$ is uncountable.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f : A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g : A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.
8. X is *uncountable* if it is not countable.

Examples. Any $n \in \omega$ is finite. ω is countably infinite. $\mathbb{R}$ is uncountable.

Theorem.

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f\colon A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g\colon A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.
8. X is *uncountable* if it is not countable.

Examples. Any $n \in \omega$ is finite. ω is countably infinite. $\mathbb{R}$ is uncountable.

Theorem. (Cantor-Bernstein-Schröder)

Equipotence, Finiteness, Countability

1. $|A| = |B|$ means there is a bijection $f: A \to B$. We read this "The cardinality of A is equal to the cardinality of B". When $|A| = |B|$ we say that A and B are *equipotent* (= "equal strength").
2. $|A| \leq |B|$ means there is an injective map $g: A \to B$.
3. $|A| < |B|$ means $|A| \leq |B|$, but $|A| \neq |B|$.
4. X is *finite* if it is equipotent with a natural number.
5. X is *infinite* if it is not finite.
6. X is *countably infinite* if it is equipotent with ω.
7. X is *countable* if it is finite or countably infinite.
8. X is *uncountable* if it is not countable.

Examples. Any $n \in \omega$ is finite. ω is countably infinite. $\mathbb{R}$ is uncountable.

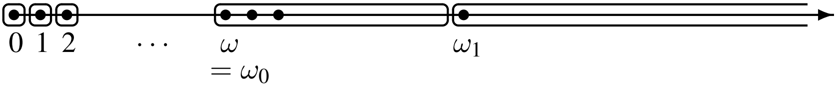
Theorem. (Cantor-Bernstein-Schröder) If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Initial ordinals

Initial ordinals

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers

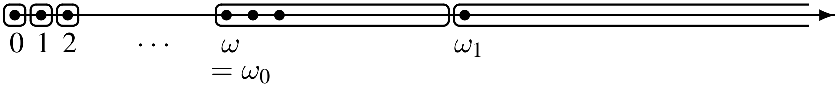


The key features of this figure are

Initial ordinals

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers



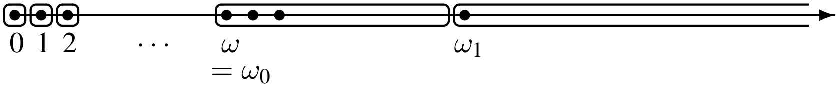
The key features of this figure are

1. Equipotence classes are intervals.

Initial ordinals

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers



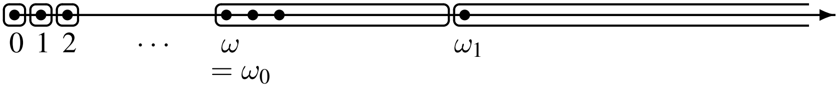
The key features of this figure are

1. Equipotence classes are intervals.

Initial ordinals

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers



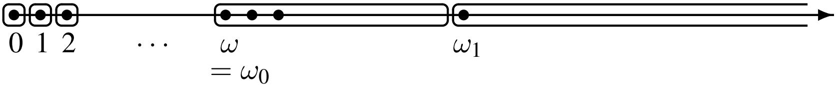
The key features of this figure are

1. Equipotence classes are intervals. The classes of natural numbers are singletons.

Initial ordinals

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers



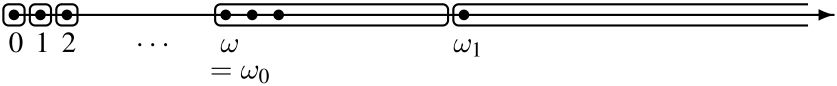
The key features of this figure are

1. Equipotence classes are intervals. The classes of natural numbers are singletons.
2. Every equipotence class has a least element.

Initial ordinals

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers



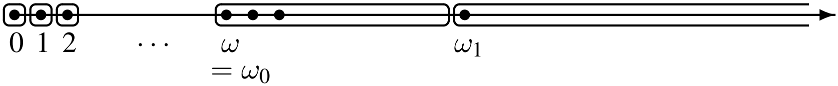
The key features of this figure are

1. Equipotence classes are intervals. The classes of natural numbers are singletons.
2. Every equipotence class has a least element.

Initial ordinals

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers



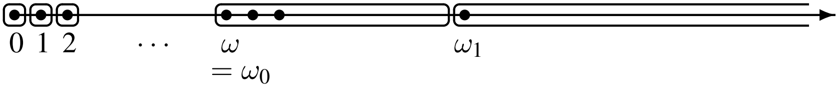
The key features of this figure are

1. Equipotence classes are intervals. The classes of natural numbers are singletons.
2. Every equipotence class has a least element. (Such elements are called *initial ordinals*.)

Initial ordinals

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers



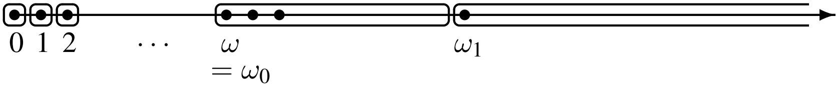
The key features of this figure are

1. Equipotence classes are intervals. The classes of natural numbers are singletons.
2. Every equipotence class has a least element. (Such elements are called *initial ordinals*.)
3. For every equipotence class, there is a strictly larger class.

Initial ordinals

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers



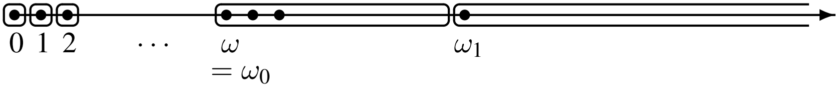
The key features of this figure are

1. Equipotence classes are intervals. The classes of natural numbers are singletons.
2. Every equipotence class has a least element. (Such elements are called *initial ordinals*.)
3. For every equipotence class, there is a strictly larger class.

Initial ordinals

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers



The key features of this figure are

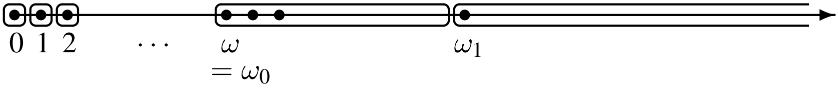
1. Equipotence classes are intervals. The classes of natural numbers are singletons.
2. Every equipotence class has a least element. (Such elements are called *initial ordinals*.)
3. For every equipotence class, there is a strictly larger class.

To measure size, we pick one ordinal from each equipotence class. Since each class has a least element, that one is the natural choice.

Initial ordinals

It follows from the CBS Theorem that equipotence classes of ordinals fall into intervals, as the next figure indicates.

Equipotence classes of ordinal numbers



The key features of this figure are

1. Equipotence classes are intervals. The classes of natural numbers are singletons.
2. Every equipotence class has a least element. (Such elements are called *initial ordinals*.)
3. For every equipotence class, there is a strictly larger class.

To measure size, we pick one ordinal from each equipotence class. Since each class has a least element, that one is the natural choice.

Cardinal numbers

Cardinal numbers

Definition.

Cardinal numbers

Definition. A **cardinal number** is an initial ordinal.

Cardinal numbers

Definition. A **cardinal number** is an initial ordinal.

When discussing cardinals, it is common to use the symbols $\aleph_0, \aleph_1, \aleph_2$ in place of $\omega_0, \omega_1, \omega_2$, ETC.

Cardinal numbers

Definition. A **cardinal number** is an initial ordinal.

When discussing cardinals, it is common to use the symbols $\aleph_0, \aleph_1, \aleph_2$ in place of $\omega_0, \omega_1, \omega_2$, ETC. $\aleph$ (aleph) is the first letter of the Hebrew alphabet.

Cardinal numbers

Definition. A **cardinal number** is an initial ordinal.

When discussing cardinals, it is common to use the symbols $\aleph_0, \aleph_1, \aleph_2$ in place of $\omega_0, \omega_1, \omega_2$, ETC. $\aleph$ (aleph) is the first letter of the Hebrew alphabet. We read $\aleph_0$ as "aleph zero" or "aleph naught".

Cardinal numbers

Definition. A **cardinal number** is an initial ordinal.

When discussing cardinals, it is common to use the symbols $\aleph_0, \aleph_1, \aleph_2$ in place of $\omega_0, \omega_1, \omega_2$, ETC. $\aleph$ (aleph) is the first letter of the Hebrew alphabet. We read $\aleph_0$ as "aleph zero" or "aleph naught". The first few cardinals are $0, 1, 2, \ldots, \aleph_0, \aleph_1, \ldots$.

Cardinal numbers

Definition. A **cardinal number** is an initial ordinal.

When discussing cardinals, it is common to use the symbols $\aleph_0, \aleph_1, \aleph_2$ in place of $\omega_0, \omega_1, \omega_2$, ETC. $\aleph$ (aleph) is the first letter of the Hebrew alphabet. We read $\aleph_0$ as "aleph zero" or "aleph naught". The first few cardinals are $0, 1, 2, \ldots, \aleph_0, \aleph_1, \ldots$.

We can refine the Well Ordering Theorem to say:

Cardinal numbers

Definition. A **cardinal number** is an initial ordinal.

When discussing cardinals, it is common to use the symbols $\aleph_0, \aleph_1, \aleph_2$ in place of $\omega_0, \omega_1, \omega_2$, ETC. $\aleph$ (aleph) is the first letter of the Hebrew alphabet. We read $\aleph_0$ as "aleph zero" or "aleph naught". The first few cardinals are $0, 1, 2, \ldots, \aleph_0, \aleph_1, \ldots$.

We can refine the Well Ordering Theorem to say:

Theorem.

Cardinal numbers

Definition. A **cardinal number** is an initial ordinal.

When discussing cardinals, it is common to use the symbols $\aleph_0, \aleph_1, \aleph_2$ in place of $\omega_0, \omega_1, \omega_2$, ETC. $\aleph$ (aleph) is the first letter of the Hebrew alphabet. We read $\aleph_0$ as “aleph zero” or “aleph naught”. The first few cardinals are $0, 1, 2, \ldots, \aleph_0, \aleph_1, \ldots$.

We can refine the Well Ordering Theorem to say:

Theorem. Every set can be enumerated by a unique cardinal number.

Definition. A **cardinal number** is an initial ordinal.

When discussing cardinals, it is common to use the symbols $\aleph_0, \aleph_1, \aleph_2$ in place of $\omega_0, \omega_1, \omega_2$, ETC. $\aleph$ (aleph) is the first letter of the Hebrew alphabet. We read $\aleph_0$ as "aleph zero" or "aleph naught". The first few cardinals are $0, 1, 2, \ldots, \aleph_0, \aleph_1, \ldots$.

We can refine the Well Ordering Theorem to say:

Theorem. Every set can be enumerated by a unique cardinal number. (For every set X, there is a unique cardinal κ for which there is a bijection $f\colon \kappa \to X$.)

Cantor's Theorem

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality.

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.
If X is a set, then $|X| < |\mathcal{P}(X)|$.

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.
If X is a set, then $|X| < |\mathcal{P}(X)|$.

Theorem.

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.
If X is a set, then $|X| < |\mathcal{P}(X)|$.

Theorem.
$|\mathbb{R}| \leq |(0,1)| \leq |\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.
If X is a set, then $|X| < |\mathcal{P}(X)|$.

Theorem.
$|\mathbb{R}| \leq |(0,1)| \leq |\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Corollary.

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.
If X is a set, then $|X| < |\mathcal{P}(X)|$.

Theorem.
$|\mathbb{R}| \leq |(0,1)| \leq |\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Corollary.
$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.
If X is a set, then $|X| < |\mathcal{P}(X)|$.

Theorem.
$|\mathbb{R}| \leq |(0,1)| \leq |\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Corollary.
$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

Notation.

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.
If X is a set, then $|X| < |\mathcal{P}(X)|$.

Theorem.
$|\mathbb{R}| \leq |(0,1)| \leq |\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Corollary.
$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

Notation.

1. For $k \in \mathbb{N}$, $|A| = k$ means $|A| = |k|$.

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.
If X is a set, then $|X| < |\mathcal{P}(X)|$.

Theorem.
$|\mathbb{R}| \leq |(0,1)| \leq |\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Corollary.
$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

Notation.

1. For $k \in \mathbb{N}$, $|A| = k$ means $|A| = |k|$.

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.
If X is a set, then $|X| < |\mathcal{P}(X)|$.

Theorem.
$|\mathbb{R}| \leq |(0,1)| \leq |\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Corollary.
$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

Notation.

1. For $k \in \mathbb{N}$, $|A| = k$ means $|A| = |k|$. This means that there is a bijection $f\colon k \to A$.

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.
If X is a set, then $|X| < |\mathcal{P}(X)|$.

Theorem.
$|\mathbb{R}| \leq |(0, 1)| \leq |\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Corollary.
$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

Notation.

1. For $k \in \mathbb{N}$, $|A| = k$ means $|A| = |k|$. This means that there is a bijection $f\colon k \to A$. We say "A has k elements".

Cantor's Theorem

The CBS Theorem helps us show two sets have the same cardinality. The following theorem helps us show two sets have different cardinality.

Cantor's Theorem.
If X is a set, then $|X| < |\mathcal{P}(X)|$.

Theorem.
$|\mathbb{R}| \leq |(0,1)| \leq |\mathcal{P}(\mathbb{N})| \leq |\mathbb{R}|$.

Corollary.
$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| = |\mathbb{R}|$.

Notation.

1. For $k \in \mathbb{N}$, $|A| = k$ means $|A| = |k|$. This means that there is a bijection $f\colon k \to A$. We say "A has k elements".
2. Similarly, for any other cardinal κ, $|A| = \kappa$ means $|A| = |\kappa|$, which means that there is a bijection $f\colon \kappa \to A$. We say "A has κ elements" or "A has κ-many elements".

The Continuum Hypothesis

The Continuum Hypothesis

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

The Continuum Hypothesis

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

Continuum Hypothesis. (Cantor) $|\mathbb{R}| = \aleph_1$.

The Continuum Hypothesis

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

Continuum Hypothesis. (Cantor) $|\mathbb{R}| = \aleph_1$.

Theorem. (Gödel, 1940)

The Continuum Hypothesis

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

Continuum Hypothesis. (Cantor) $|\mathbb{R}| = \aleph_1$.

Theorem. (Gödel, 1940) If ZFC is consistent, then so is ZFC + CH.

The Continuum Hypothesis

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

Continuum Hypothesis. (Cantor) $|\mathbb{R}| = \aleph_1$.

Theorem. (Gödel, 1940) If ZFC is consistent, then so is ZFC + CH.

Theorem. (Cohen, 1963)

The Continuum Hypothesis

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

Continuum Hypothesis. (Cantor) $|\mathbb{R}| = \aleph_1$.

Theorem. (Gödel, 1940) If ZFC is consistent, then so is ZFC + CH.

Theorem. (Cohen, 1963) If ZFC is consistent, then so is ZFC + ¬ CH.

The Continuum Hypothesis

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

Continuum Hypothesis. (Cantor) $|\mathbb{R}| = \aleph_1$.

Theorem. (Gödel, 1940) If ZFC is consistent, then so is ZFC + CH.

Theorem. (Cohen, 1963) If ZFC is consistent, then so is ZFC + ¬ CH.

Conclusion.

The Continuum Hypothesis

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

Continuum Hypothesis. (Cantor) $|\mathbb{R}| = \aleph_1$.

Theorem. (Gödel, 1940) If ZFC is consistent, then so is ZFC + CH.

Theorem. (Cohen, 1963) If ZFC is consistent, then so is ZFC + ¬ CH.

Conclusion. If the axioms of set theory are consistent, then they are not strong enough to decide whether $|\mathbb{R}| = \aleph_1$.

The Continuum Hypothesis

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

Continuum Hypothesis. (Cantor) $|\mathbb{R}| = \aleph_1$.

Theorem. (Gödel, 1940) If ZFC is consistent, then so is ZFC + CH.

Theorem. (Cohen, 1963) If ZFC is consistent, then so is ZFC + $\neg$ CH.

Conclusion. If the axioms of set theory are consistent, then they are not strong enough to decide whether $|\mathbb{R}| = \aleph_1$. In fact, we also cannot decide whether $|\mathbb{R}| = \aleph_2$ or $\aleph_3$ or $\aleph_4$ or $\ldots$,

The Continuum Hypothesis

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

Continuum Hypothesis. (Cantor) $|\mathbb{R}| = \aleph_1$.

Theorem. (Gödel, 1940) If ZFC is consistent, then so is ZFC + CH.

Theorem. (Cohen, 1963) If ZFC is consistent, then so is ZFC + $\neg$ CH.

Conclusion. If the axioms of set theory are consistent, then they are not strong enough to decide whether $|\mathbb{R}| = \aleph_1$. In fact, we also cannot decide whether $|\mathbb{R}| = \aleph_2$ or $\aleph_3$ or $\aleph_4$ or ..., but we do know that $|\mathbb{R}| \neq \aleph_\omega$.

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

Continuum Hypothesis. (Cantor) $|\mathbb{R}| = \aleph_1$.

Theorem. (Gödel, 1940) If ZFC is consistent, then so is ZFC + CH.

Theorem. (Cohen, 1963) If ZFC is consistent, then so is ZFC + $\neg$ CH.

Conclusion. If the axioms of set theory are consistent, then they are not strong enough to decide whether $|\mathbb{R}| = \aleph_1$. In fact, we also cannot decide whether $|\mathbb{R}| = \aleph_2$ or $\aleph_3$ or $\aleph_4$ or . . ., but we do know that $|\mathbb{R}| \neq \aleph_\omega$. (We cannot decide $|\mathbb{R}| = \aleph_{\omega+1}, \aleph_{\omega+2}, \aleph_{\omega+3}, \ldots$.)

The Continuum Hypothesis

We know that $|\mathbb{N}| = \aleph_0 < |\mathbb{R}|$, so $|\mathbb{R}| = \aleph_\alpha$ for some $\alpha > 0$.

Continuum Hypothesis. (Cantor) $|\mathbb{R}| = \aleph_1$.

Theorem. (Gödel, 1940) If ZFC is consistent, then so is ZFC + CH.

Theorem. (Cohen, 1963) If ZFC is consistent, then so is ZFC + ¬ CH.

Conclusion. If the axioms of set theory are consistent, then they are not strong enough to decide whether $|\mathbb{R}| = \aleph_1$. In fact, we also cannot decide whether $|\mathbb{R}| = \aleph_2$ or $\aleph_3$ or $\aleph_4$ or $\ldots$, but we do know that $|\mathbb{R}| \neq \aleph_\omega$. (We cannot decide $|\mathbb{R}| = \aleph_{\omega+1}, \aleph_{\omega+2}, \aleph_{\omega+3}, \ldots$.)