Section 8.5
1. Prove the following generalization of the Burnside $p - q$ Theorem: A finite group with a nilpotent subgroup of prime-power index is soluble.

Proof. We proceed by induction on $|G|$. First, note that if $|G| = 1$, then G is finite, and G is a nilpotent subgroup of prime-power index, and G is solvable, so the base case holds. Next, suppose that the result holds for all groups of strictly smaller order than $|G|$, and G be finite with a nilpotent subgroup H of prime-power index. We show that it suffices to prove that G is not simple. To this end, suppose that G has a normal subgroup N such that $\{1\} \neq N$ and $N \neq G$. We have that G/N is finite with $|G/N| < |G|$. Further, HN/N is a subgroup of G, and by the second isomorphism theorem, $HN/N \cong H/H \cap N$. $H/H \cap N$ is nilpotent since it is a quotient of a nilpotent group, and so HN/N is nilpotent. By the third isomorphism theorem, we have that $[G/N : HN/N] = [G : HN]$, which is a prime power since $H \leq HN$, so $[G : H] = [G : HN][HN : H]$, and $[G : H]$ is a prime power. Thus G/N is a group of strictly smaller order than G that contains a nilpotent subgroup of prime power order. By induction, this implies that G/N is solvable. Further, $H \cap N$ is nilpotent since $H \cap N \leq N$, and $[N : H \cap N] = [HN : H]$, which is a prime power, so again the induction hypothesis gives that N is solvable. But then since G/N and N are solvable, so is G, which finishes the proof. Thus it remains only to show that G has a proper normal subgroup. For this, we note that since H is nilpotent, there exists $1 \neq h \in Z(H)$. If $h \in Z(G)$, then $Z(G)$ is nontrivial, and since $Z(G) \lhd G$, we have that G is solvable by the preceding argument. On the other hand, if $h \notin Z(G)$, then by the orbit-stabilizer theorem, we have that $|C(h)| = [G : \mathrm{Stab}(h)]$, where $C(h)$ is the conjugacy class of h. Since $\mathrm{Stab}(h) \geq H$, we have that $[G : \mathrm{Stab}(h)]$ is a prime power. Since $h \notin Z(G)$, $[G : \mathrm{Stab}(h)] \neq 1$. Thus h is an element whose conjugacy class has size p^m for $m > 0$, and by Proposition 8.5.2 in Robinson, G is not simple. Thus G has a proper normal subgroup, which again finishes the proof by the preceding argument. □