Dan Lyness
Calum Shearer
Andrew Doumont

Group Theory
Assignment 3

7. Prove that a finite nonabelian simple group has no irrep of degree 2.

*Proof.* Let $G$ be a finite nonabelian simple group, and suppose for a contradiction that there exists an irreducible representation $\rho$ of degree 2. Since $G$ is simple, $\mathrm{Ker}(\rho) = \{0\}$ or $G$, but $\rho$ is not the trivial representation so we must have $\mathrm{Ker}(\rho) = \{0\}$. Hence $\rho : G \hookrightarrow GL(2, \mathbb{C})$ is an injection. By property (38) of the handout, $2 \mid |G|$, so by Cauchy's theorem there exists an element $g \in G$ of order 2. Hence $\rho(g)^2 = I$ so $\rho(g)$ satisfies the equation $x^2 - 1 = 0$, so can only have eigenvalues $\pm 1$. $\rho$ is injective, so $\rho(g) \neq I$. Since $\rho(g)$ is diagonalisable, we can pick a basis so that $\rho$ takes one of the following forms:

1. $\rho(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

2. $\rho(g) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

1. Suppose $\rho(g) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then consider the composition $\mathrm{Det} \circ \rho : G \to \mathbb{C}^\times$. $\mathrm{Det}(\rho(g)) = -1$, so $\mathrm{Det} \circ \rho$ is a nontrivial homomorphism, hence must be injective since $G$ is simple. Hence $\mathrm{Det} \circ \rho : G \hookrightarrow \mathbb{C}^\times$ is an embedding of $G$ into an abelian group, so $G$ must be abelian, a contradiction.

2. Suppose $\rho(g) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = -I$. But this matrix commutes with all of $GL(2, \mathbb{C})$, so $\rho(g) \in Z(GL(2, \mathbb{C}))$. Since $\rho$ was injective, this implies that $g \in Z(G)$, so $Z(G)$ is a nontrivial normal subgroup of $G$. Since $G$ is simple, we must have that $Z(G) = G$, so $G$ is abelian which is a contradiction.

$\square$