Rebecca DeLand
Bob Kuo
Matt Watson

5. Let G be a finite group. Suppose that every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 1$ vanishes off of G'. Show that each nonidentity coset of G' is a conjugacy class.

Proof. Let G be a finite group and suppose that every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 1$ vanishes off of G'. Additionally, suppose that $g \notin G'$ and $gh \in gG'$. If χ is an irreducible linear character (so $\chi(1) = 1$) and ρ is the associated representation, then there is a representation $\rho_{G'}$ on G/G' such that $\rho = \rho_{G'} \circ \nu$ by inflation. Then

$$\rho(gh) = \rho_{G'}(\nu(gh)) = \rho_{G'}(\nu(g)) = \rho(g).$$

Therefore

$$\chi(gh) = \mathrm{tr}(\rho(gh)) = \mathrm{tr}(\rho(g)) = \chi(g).$$

If χ is an irreducible nonlinear character, then $\chi(1) > 1$. Since $g \notin G'$ and χ vanishes off of G', $\chi(g) = 0$. Similarly, since $gG' \cap G' = \emptyset$ and $gh \in gG'$, $\chi(gh) = 0$. Therefore $\chi(g) = \chi(gh)$ for every $\chi \in \mathrm{Irr}(G)$.

Recall that one property of characters is that they are class functions and the columns (conjugacy classes) of a character table are orthogonal, we must have gh is in the same conjugacy class as g since $\chi(g) = \chi(gh)$ for all $\chi \in \mathrm{Irr}(G)$. Therefore each nonidentity coset of G' is a conjugacy class. $\square$