Breeann Wilson
Justin Willson

8. Show that if an abelian group A contains elements $a_0, a_1, a_2, \ldots$ satisfying

(i) $a_0 \neq 0$, and

(ii) $pa_{i+1} = a_i$ holds for all i,

then the subgroup of A generated by these elements is isomorphic to $\mathbb{Z}_{p^\infty}$.

Proof. We do not believe the statement is correct as written. Consider the elements $\{1/2, 1/4, 1/8, \ldots\}$ in $\mathbb{Q}$. Then these elements satisfy the given relations for $p = 2$, but the subgroup they generate contains elements of infinite order, which $\mathbb{Z}_{p^\infty}$ does not have. We believe the conditions should be amended to include

(iii) $pa_0 = 0$

In order to prove this amended fact, we will first prove a statement about quotients of $\mathbb{Z}_{p^\infty}$. We claim that any quotient of $\mathbb{Z}_{p^\infty}$ is isomorphic to $\mathbb{Z}_{p^\infty}$. To that end, let I be a proper subgroup of $\mathbb{Z}_{p^\infty}$ and consider the presentation of $\mathbb{Z}_{p^\infty}$ given by

$$\mathbb{Z}_{p^\infty} = \langle g_0, g_1, \ldots \mid g_0^p = 1, g_{i+1}^p = g_i \rangle.$$

If I is given by $\langle g_0, g_1, \ldots, g_n \rangle = \langle g_n \rangle$ for some n, then we've proven our claim since the map $\phi : \mathbb{Z}_{p^\infty}/I \to \mathbb{Z}_{p^\infty}$ given by $\phi([g_i]) = g_{i-n}$ for each $i > n$, where $[g_i]$ is the equivalence class of g_i under the quotient map, gives an isomorphism between the two groups. Thus, we have reduced showing that every quotient of $\mathbb{Z}_{p^\infty}$ by a proper subgroup is isomorphic to $\mathbb{Z}_{p^\infty}$ to showing that every proper subgroup of $\mathbb{Z}_{p^\infty}$ is given by $\langle g_n \rangle$ for some n.

To that end, let I be a proper subgroup of $\mathbb{Z}_{p^\infty}$ and let b be in I. Then we can write

$$b = k_1 a_{i_1} + \ldots + k_1 a_{i_n}$$

with $i_1 < i_2 < \ldots < i_n$ and $0 < k_j < p$ for $1 \leq j \leq n$. This is because the a_i generate $\mathbb{Z}_{p^\infty}$ and because if k_j is greater than p, we can factor out powers of p and rewrite our element in terms of a_i for some $i < j$. We can further rewrite

$$b = (k_1 p_{i_n - i_1} + \ldots + k_n) a_{i_n}$$

by writing $a_{i_j} = p^{i_n - i_j} a_{i_n}$ for any $i_j < i_n$. This tells us that b is in $I \cap \langle a_{i_n} \rangle$. Further, since k_n is not divisible by p and is not equal to zero, we see that b is not in $I \cap \langle a_{i_n - 1} \rangle$. This shows that $I \cap \langle a_{i_n} \rangle = \langle a_{i_n} \rangle$ since $\langle a_{i_n} \rangle$ is a p-group, hence, its subgroup lattice is just a chain.

Now, either I is finite or it isn't. If it is finite, I can choose a $b = k_1 a_{i_1} + \ldots + k_1 a_{i_n}$ with i_n maximal. From here, we see $I \cap \langle a_{i_n} \rangle = \langle a_{i_n} \rangle$. Even more than this, we see $I \cap \langle a_j \rangle = \langle a_{i_n} \rangle$ for any $j > i_n$. Thus, $I = \langle a_{i_n} \rangle$ as it cannot contain any other generators. Next, if I is not finite, then I contains every generator since for each m, we can choose a $b = k_1 a_{i_1} + \ldots + k_1 a_{i_n}$ for some $i_n > m$, so we get that $a_m \in \langle a_{i_n} \rangle = I \cap \langle a_{i_n} \rangle \subset I$. Hence, $I = \mathbb{Z}_{p^\infty}$ and is therefore not a proper subgroup. Thus we have shown that every quotient of $\mathbb{Z}_{p^\infty}$ by a proper subgroup is isomorphic to $\mathbb{Z}_{p^\infty}$.

Now, to prove the original fact, consider the universal property of presentations. That is, if $a_0, a_1, \ldots \in A$ satisfy the same relations as the generators of $\mathbb{Z}_{p^\infty}$, then the subgroup generated by these elements is isomorphic to a quotient of $\mathbb{Z}_{p^\infty}$. We know a_0 is non-zero, so $\langle a_0, a_1, \ldots \rangle$ is non-zero. Thus, it must be a quotient of $\mathbb{Z}_{p^\infty}$ by some proper subgroup; hence, it must be isomorphic to $\mathbb{Z}_{p^\infty}$.

$\square$