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7. Show that TFAE for nonzero abelian groups.

(i) D is divisible.

(ii) D has no maximal subgroups.

(iii) D has no finite nonzero quotient.

Proof. (i) $\implies$ (iii): Assume $\neg$ (iii), namely that D has a finite non-zero quotient, D/H of cardinality n for some $n \in \mathbb{N}$. Then the multiplication by n map on D/H is the zero map: Hence any non-zero element of D/H is *not* divisible by n. Hence D/H is not divisible. The quotient of a divisible group is also divisible, so D is *not* divisible, i.e. $\neg$ (i).

(iii) $\implies$ (i): Suppose that D has no finite non-zero quotients. Then we wish to show that $pD = D$ for all primes p, as this implies divisibility. Pick any prime p and consider the quotient group D/pD. Then this is an abelian p-group, with all elements having order dividing p. Indeed, given $[x] = x + pD$ with $x \in D$:

$$p[x] = [px] = pD = 0$$

So D/pD is a group of exponent 0 or p, and therefore is (or can be endowed with the structure of) an $\mathbb{F}_p$ vector space. It therefore follows that unless $D/pD = 0$, it has a quotient that is 1-dimensional, and so said quotient is isomorphic to $\mathbb{F}_p$ (which is finite and nonzero). However, this is then also isomorphic to a quotient of D, which contradicts (iii). Thus $D/pD = 0$ and so $D = pD$. As p was an arbitrary prime, we get divisibility of D.

(iii) $\implies$ (ii): Assume $\neg$ (ii), namely that D has a maximal subgroup, M. Then by the Fourth Isomorphism Theorem for groups and the fact that D is abelian, D/M is simple and non-zero. So pick some $[a] \in D/M \setminus \{0\}$. Then, as D/M is simple, $\langle [a] \rangle = D/M$, i.e. $D/M \cong \mathbb{Z}$ or $D/M \cong \mathbb{Z}_n$ for some $n \in \mathbb{N}$. But $\mathbb{Z}$ is not simple: it has $2\mathbb{Z}$ as a proper subgroup. Hence the quotient D/M is non-zero and finite, i.e. $\neg$ (iii).

(ii) $\implies$ (iii): Assume $\neg$ (iii), namely that D has a finite non-zero quotient. Then choose $H \leq D$ such that D/H has least non-zero cardinality amongst finite non-zero quotients of D (note: such a H exists because there exists *some* non-zero finite quotient). Then if $H \subsetneq H' \subsetneq D$ for some $H' \leq D$, $0 < |D/H'| < |D/H|$, which contradicts the minimality of $|D/H|$. Hence H is a maximal subgroup of D, i.e. $\neg$ (ii).

$\square$