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5. Show that if A is an abelian group, then there is an abelian group D, unique up to isomorphism over A, such that

   (a) $A \leq D$,

   (b) D is divisible, and

   (c) if $M \leq D$, $M \neq \{0\}$, then $A \cap M \neq \{0\}$

   D is called the divisible hull of A.

   Proof. From proposition 4.1.6 (Robinson pg. 98), we know that there exists some divisible abelian group D' such that $A \leq D'$. Take a maximal subgroup $M \leq D'$ such that $A \cap M = \{0\}$ and consider $D = D'/M$. We know that such a maximal subgroup exists by Zorn's Lemma. To see this, suppose first that there is at least some subgroup disjoint from A (otherwise we are already where we want to be), then giving any chain of subgroups $B_1, B_2, \ldots$ disjoint from A, we can take the union $\cup B_i$ and we will again have a subgroup disjoint from A. This union is an upper bound on the sequence $B_1, B_2, \ldots$, so we can apply Zorn's Lemma and see that such a maximal disjoint subgroup exists.

   Now, we can see that A is isomorphic to a subgroup of D since we can take the map $\iota : A \to D'/M$ given by $a \mapsto a + M$ and see that this map is a homomorphism since $\iota(a+b) = (a+b)+M = a+M+b+M = \iota(a)+\iota(b)$. Additionally, it is injective since if

   $$\iota(a) = a + M = b + M = \iota(b)$$

   then $a - b \in M$ but a, b are in A so that $a - b$ must be in the intersection of M and A. But this shows that $a - b = 0$, so that $a = b$. So, we can identify A with its isomorphic image in D. There can be no nontrivial subgroup disjoint from A in D because if there were this would imply that M was not maximal. To see this, observe that if D contained a nonzero subgroup disjoint from $i(A)$, then the subgroup would correspond to a proper extension of M in D' disjoint from A, which would contradict that M is maximal such that M is disjoint from A.

   Additionally, we know that quotients of divisible groups are divisible, so that D is divisible. So, D satisfies our three properties and now we just want to show that D is unique up to isomorphism over A.

   Suppose D_1, D_2 are two such divisible abelian groups. We know from the fact that D_2 is injective that we can extend the inclusion $A \to D_1$ to a map $\varphi : D_1 \to D_2$. That is we have the following commutative diagram

   $$\begin{tikzcd} A \arrow[r, hook] \arrow[d, hook] & D_1 \arrow[dl, dashed, "\varphi"] \\ D_2 & \end{tikzcd}$$

   where φ must be injective. φ must be injective because otherwise the kernel of φ will be a subgroup of D_1 disjoint from A, and this contradicts that D_1 was chosen to be a group for which this is not possible. This is because no element of A can be part of the kernel since φ is an extension of the inclusion of A into D_1. But, now we have a short exact sequence

$$0 \longrightarrow D_1 \xrightarrow{\varphi} D_2 \xrightarrow{\overline{\varphi}} D_2/D_1 \longrightarrow 0$$

and since D_1 is injective, we know that this SES splits. Therefore, we have

$$D_2 \cong D_1 \oplus D_2/D_1.$$

But $A \leq D_1$ and so $D_2/D_1 \cap A = \{0\}$. But, by assumptions on D_2 this implies that D_2/D_1 must be trivial, and so $D_2 \cong D_1$ and φ is an isomorphism that fixes A. Therefore, D is unique up to isomorphism over A.

$\square$