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4. Show that a quotient of an injective R-module need not be injective.

Proof. We claim that $\mathbb{Z}_4$ is an injective $\mathbb{Z}_4$-module but the quotient $\mathbb{Z}_4/(2) \cong \mathbb{Z}_2$ is not an injective $\mathbb{Z}_4$-module. Recall the following result:

> Baer's Criterion. Let R be a ring with 1 and let J be a left R-module. Then J is injective if and only if for every left ideal $I \subseteq R$ and every $f \in \mathrm{Hom}_R(I, J)$, there exists $\overline{f} \in \mathrm{Hom}_R(R, J)$ such that $\overline{f}|_I = f$.

To show that $\mathbb{Z}_4$ is an injective $\mathbb{Z}_4$-module, first note that $\mathbb{Z}_4$ is a ring with 1 and we can view this ring as a module over itself where the action is given by left multiplication. Let I be a left ideal of $\mathbb{Z}_4$; then I is equal to (0), $2\mathbb{Z}_4 \cong \mathbb{Z}_2$, or $\mathbb{Z}_4$. Suppose $f : I \to \mathbb{Z}_4$ is a $\mathbb{Z}_4$-module homomorphism. If $I = (0)$ then we can extend the map f to the zero homomorphism $\overline{f} : \mathbb{Z}_4 \to \mathbb{Z}_4$ where $x \mapsto 0$. If $I = \mathbb{Z}_4$, then we take $\overline{f}$ to be exactly equal to the map f. Now suppose $I = \mathbb{Z}_2$. Again, if $f : \mathbb{Z}_2 \to \mathbb{Z}_4$ is the zero map, then we can take $\overline{f} : \mathbb{Z}_4 \to \mathbb{Z}_4$ to be the zero map $x \mapsto 0$. Alternatively, if $f : \mathbb{Z}_2 \to \mathbb{Z}_4$ is nonzero, then we must have $0 \mapsto 0$ and $1 \mapsto 2$. Define $\overline{f} : \mathbb{Z}_4 \to \mathbb{Z}_4$ via $x \mapsto 2x$; then $\overline{f}$ is a $\mathbb{Z}_4$-module homomorphism and $\overline{f}|_{\mathbb{Z}_2} = f$. This shows that for every ideal I of $\mathbb{Z}_4$ and every $f \in \mathrm{Hom}_{\mathbb{Z}_4}(I, \mathbb{Z}_4)$, there exists $\overline{f} \in \mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_4)$ such that $\overline{f}|_I = f$. Hence by Baer's Criterion, $\mathbb{Z}_4$ is an injective $\mathbb{Z}_4$-module.

Recall that an abelian group I is injective if and only if every short exact sequence of the form $0 \to I \to B \to C \to 0$ splits; we use this fact to show that $\mathbb{Z}_2$ is not an injective $\mathbb{Z}_4$-module by first constructing such a short exact sequence. Consider the map $f : \mathbb{Z}_2 \to \mathbb{Z}_4$ defined by $f(x) = 2x$ (modulo 4) and the map $g : \mathbb{Z}_4 \to \mathbb{Z}_2$ defined by $g(x) = x$ (modulo 2). Then $f \in \mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_2, \mathbb{Z}_4)$, $g \in \mathrm{Hom}_{\mathbb{Z}_4}(\mathbb{Z}_4, \mathbb{Z}_2)$, and $\mathrm{Ker} g = \mathrm{Im} f$, meaning that we have the following short exact sequence:

$$0 \to \mathbb{Z}_2 \xrightarrow{f} \mathbb{Z}_4 \xrightarrow{g} \mathbb{Z}_2 \to 0. \tag{‡}$$

By way of contradiction, assume this short exact sequence splits. Then $\mathbb{Z}_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$; however, $\mathbb{Z}_4$ is cyclic whereas $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ is not. This is a contradiction. Thus the short exact sequence does not split and hence $\mathbb{Z}_2$ is not an injective $\mathbb{Z}_4$-module. $\square$