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Groups Homework 1

2. If R is not left Noetherian, then some direct sum of injective R modules is not injective.

PROOF: Because R is not left Noetherian, there is some strictly increasing chain $I_0 \leq I_1 \leq I_2 \leq \dots$ where I_j is a left ideal of R for $j \in \omega$. Now note that $I = \bigcup I_j$ is a left ideal of R. Viewing R as a left R-module, we see that I and I_j are submodules for $j \in \omega$. So for each $j \in \omega$, we know that R/I_j is a left R-module. Because any module is contained in an injective module we may find a left R-module M_j such that $R/I_j \subseteq M_j$ for each $j \in \omega$.

For $j \in \omega$, let $\nu_j : I \to I/I_j$ be the canonical (module) homomorphism. Note that $I/I_j \subseteq R/I_j \subseteq M_j$. Define $\nu : I \to \prod M_j$ by $\nu(x) = (\nu_0(x), \nu_1(x), \nu_2(x), \dots) = (\nu_j(x))_j$. Then ν is a homomorphism and we claim that $\nu(I) \subseteq \bigoplus M_j$. To prove this, let $x \in I$ so that $x \in I_k$ for some $k \in \omega$. Then we also have $x \in I_l$ for each $l > k$. Therefore $\nu_l(x)$ is the additive identity I_l of M_l for $l \geq k$. For readability, we will use the notation $\nu_l(x) = [0]_l$. We have

$$\nu(x) = (\nu_0(x), \dots \nu_{k-1}(x), [0]_k, [0]_{k+1}, \dots).$$

That is, $\nu(x) \in \bigoplus M_j$. Now we claim that ν cannot be extended to $\widehat{\nu} : R \to \bigoplus M_j$. This will show that $\bigoplus M_j$ is not injective by definition.

Suppose towards a contradiction that we have such an extension $\widehat{\nu}$. Then in particular we have $\widehat{\nu}(1) \in \bigoplus M_j$ which means

$$\widehat{\nu}(1) = (\pi_0 \widehat{\nu}(1), \dots, \pi_k \widehat{\nu}(1), [0]_{k+1}, [0]_{k+2}, \dots)$$

for some $k \in \omega$ where each π_i is a projection map. But for $y \in I$ we have

$$(*) \quad \widehat{\nu}(y) = \widehat{\nu}(y \cdot 1) = y \cdot \widehat{\nu}(1) = y \cdot (\pi_0 \widehat{\nu}(1), \dots, \pi_k \widehat{\nu}(1), [0]_{k+1}, [0]_{k+2}, \dots).$$

If we choose $y \in I$ such that $y \notin I_{k+1}$ (this is possible since we started with a strictly increasing chain), then we have $\nu(y) = (\nu_0(y), \dots, \nu_{k+1}(y), \dots)$ where $\nu_{k+1}(y) \neq [0]_{k+1}$. We see that this is a contradiction by observing $(*)$ and noting that $\nu(y) = \widehat{\nu}(y)$.