Andrew Doumont
Bob Kuo
Justin Willson

Problem 1.

(a) Show that a group is 2-Engel (satisfies $[x, y, y] = 1$) if and only if every 1-generated normal subgroup is abelian.

(b) Show that the following laws are consequences of $[x, y, y] = 1$:

(i) $[x, y, z] = [y, x, z]$

(ii) $[x, y, z]^3 = 1$

(iii) $[x, y, z, t] = 1$

Proposition 1. *Satisfying the law $[a, b, b] = 1$ is equivalent to $[a, b] = [b^{-1}, a]$. Hence, $[a, b]^{-1} = [a, b^{-1}]$ in a 2-Engel group.*

Proof. Notice that

$$[a, b, b] = b^{-1}a^{-1}bab^{-1}a^{-1}b^{-1}abb = [b, a]b^{-1}[a, b]b.$$

Thus $[a, b, b] = 1$ if and only if

$$\begin{aligned} b^{-1}[a, b]b &= [a, b] \\ [a, b] &= b[a, b]b^{-1} \\ [a, b] &= ba^{-1}b^{-1}abb^{-1} \\ [a, b] &= [b^{-1}, a]. \end{aligned}$$

□

Proposition 2. *A group G is 2-Engel if and only if every 1-generated normal subgroup is abelian.*

Proof. A normal subgroup generated by $x \in G$ is the subgroup of G generated by $\{y^{-1}xy : y \in G\}$. Thus, it suffices to show that x commutes with its conjugates. Observe that $xy^{-1}xy = y^{-1}xyx$ if and only if

$$\begin{aligned} xy^{-1}xy &= y^{-1}xyx \\ xy^{-1}xyx^{-1} &= y^{-1}xy \\ y^{-1}xyx^{-1} &= x^{-1}y^{-1}xy \\ [y, x^{-1}] &= [x, y]. \end{aligned}$$

Hence, the result follows from Proposition 1, with $a = y$, $b = x^{-1}$. □

Proposition 3. *Let A be the normal closure of $a \in G$, then the map $r_b(x) = [x, b]$ is an endomorphism of A for each $g \in G$. Since A is commutative, we use the notation $r_b + r_c$ to denote the map $x \mapsto r_b(x)r_c(x)$ and $-r_b$ to denote the map $x \mapsto [x, b]^{-1}$. Furthermore, this endomorphism satisfies $r_b \circ r_b = 0$ (0 denoting the map $x \mapsto 1$).*

Andrew Doumont
Bob Kuo
Justin Willson

Proof. For $x, y \in A$, notice that since $[x, b]$ is a product of conjugates of x and x^{-1} which all commute with y. Thus, $r_b(xy) = [xy, b] = [x, b]^y[x, b] = [x, b][y, b] = r_b(x) + r_b(y)$. Next, applying Proposition 1, we have $r_b(-x) = [x^{-1}, b] = [b, x^{-1}]^{-1} = [b, x] = [x, b]^{-1} = -r(b)$. The fact that $r_b \circ r_b = 0$ is just the 2-Engel condition: $r_b \circ r_b(x) = [x, b, b] = 1$. $\square$

Proposition 4. *Let $b, c \in G$. Then*

(i) $r_{b^{-1}} = -r_b$

(ii) $r_{bc} = r_c + r_b + r_c \circ r_b$

(iii) $r_b \circ r_c = -r_c \circ r_b$

Proof.

(i) This follows from Proposition 1, that $[x, b^{-1}] = [x, b]^{-1}$.

(ii) Observe that

$$[x, bc] = [x, c][x, b]^c = [x, c][x, b][x, b]^{-1}c^{-1}[x, b]c = [x, c][x, b][x, b, c] = (r_c{+}r_b{+}r_c{\circ}r_b)(x).$$

(iii) Applying (i) and (ii), we have that

$$\begin{aligned}
r_{(c^{-1}b^{-1})} \circ r_{bc} =&(r_{b^{-1}} + r_{c^{-1}} + r_{b^{-1}} \circ r_{c^{-1}}) \circ (r_c + r_b + r_c \circ r_b)\\
=&r_{b^{-1}} \circ r_c + r_{b^{-1}} \circ r_b + r_{b^{-1}} \circ r_c \circ r_b + r_{c^{-1}} \circ r_c + r_{c^{-1}} \circ r_b + r_{c^{-1}} \circ r_c \circ r_b\\
&+ r_{b^{-1}} \circ r_{c^{-1}} \circ r_c + r_{b^{-1}} \circ r_{c^{-1}} \circ r_b + r_{b^{-1}} \circ r_{c^{-1}} \circ r_c \circ r_b\\
=& - r_b \circ r_c - r_b \circ r_b - r_b \circ r_c \circ r_b - r_c \circ r_c - r_c \circ r_b - r_c \circ r_c \circ r_b\\
&- r_b \circ (-r_c \circ r_c) - r_b \circ (-r_c \circ r_b) - r_c \circ (-r_c \circ r_c \circ r_b)\\
=& - r_b \circ r_c - r_b \circ r_c \circ r_b - r_c \circ r_b + r_b \circ r_c \circ r_b\\
=& - r_b \circ r_c - r_c \circ r_b.
\end{aligned}$$

Note that the fourth equality follows from the third because $r_a \circ r_a = 0$ for arbitrary $a \in A$. That is, terms with an endomorphism composed with itself disappears. Now, $r_{(c^{-1}b^{-1})} \circ r_{bc} = r_{(bc)^{-1}} \circ r_{bc} = -r_{bc} \circ r_{bc} = 0$. Hence $0 = -r_b \circ r_c - r_c \circ r_b$.

$\square$

Proposition 5. *The following are identities in a 2-Engel group G:*

(i) $[x, y, z] = [y, z, x]$

(ii) $[x, y, z]^3 = 1$

(iii) $[x, y, z, t] = 1$

Andrew Doumont
Bob Kuo
Justin Willson

Proof.

(i) Using Proposition 4 (iii), we have $r_z \circ r_y(x) = -r_y \circ r_z(x)$. Expanding this, we have $[x, y, z] = [x, z, y]^{-1}$. Now, observe that

$$[x, z, y]^{-1} = ([x, z]^{-1}y^{-1}[x, z]y)^{-1} = y^{-1}[x, z]^{-1}y[x, z]$$

Then, since $[x, z]$ commutes with conjugates of its inverse, we can re-write

$$y^{-1}[x, z]^{-1}y[x, z] = [x, z]y^{-1}[x, z]^{-1}y = [z, x]^{-1}y^{-1}[z, x]y = [z, x, y].$$

That is, $[x, y, z] = [z, x, y]$. Applying the same argument again to $[z, x, y]$ gives $[x, y, z] = [y, z, x]$.

(ii) Using (i), notice that $[x, y^{-1}, z] = [z, x, y^{-1}] = [z, x]^{-1}y[z, x]y^{-1}$ is a product of y^{-1} and a conjugate of y. Thus, $[x, y^{-1}, z]$ commutes with y so $[x, y^{-1}, z]^y = [x, y^{-1}, z]$. Furthermore, again using that elements commute with conjugates of their inverses,

$$[z, x]^{-1}y[z, x]y^{-1} = y[z, x]y^{-1}[z, x]^{-1} = [z, x]y^{-1}[z, x]^{-1}y = [x, z, y] = [x, y, z]^{-1}.$$

Thus, we have $[x, y^{-1}, z] = [x, y, z]^{-1}$ so applying the Hall-Witt identity, we have

$$\begin{aligned}
1 &= [x, y^{-1}, z]^y[y, z^{-1}, x]^z[z, x^{-1}, y]^x \\
1 &= [x, y^{-1}, z][y, z^{-1}, x][z, x^{-1}, y] \\
1 &= [x, y, z]^{-1}[y, z, x]^{-1}[z, x, y]^{-1} \\
1 &= [x, y, z]^{-3}.
\end{aligned}$$

Then, multiplying through by $[x, y, z]^3$ gives the desired result.

(iii) Note that $[x, y, z, t] = r_t \circ r_z \circ r_y(x)$, so it suffices to show that $r_d \circ r_c \circ r_b = 0$ for any $b, c, d \in G$. Applying Proposition 4 (iii), we have $r_b \circ r_{cd} = -r_{cd} \circ r_b$. So applying part (ii) of the same proposition to r_{cd}, we have

$$\begin{aligned}
r_b \circ (r_d + r_c + r_d \circ r_c) &= -(r_d + r_c + r_d \circ r_c) \circ r_b \\
r_b \circ r_d + r_b \circ r_c + r_b \circ r_d \circ r_c &= -r_d \circ r_b - r_c \circ r_b - r_d \circ r_c \circ r_b \\
r_b \circ r_d + r_b \circ r_c + r_b \circ r_d \circ r_c &= -(-r_b \circ r_d) - (-r_b \circ r_c) - r_d \circ r_c \circ r_b \\
r_b \circ r_d \circ r_c &= -r_d \circ r_c \circ r_b \\
2r_d \circ r_c \circ r_b &= 0.
\end{aligned}$$

The last equality is the result of $r_b \circ r_d \circ r_c = -(r_d \circ r_b) \circ r_c = -r_d \circ (-r_c \circ r_b) = r_d \circ r_c \circ r_b$. We also have $3r_d \circ r_c \circ r_b = 0$ since $3r_d \circ r_c \circ r_b(x) = [x, b, c, d]^3 = [[x, b], c, d]^3 = 0$ by part (ii). Thus, $2r_d \circ r_c \circ r_b = 3r_d \circ r_c \circ r_b$ from which we get $r_d \circ r_c \circ r_b = 0$.

□