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Group Theory
Assignment 1

3. Show that the retract of an injective R-module is injective.

Proof. Let Q be an injective R-module, and E be a retract of Q. Then there exist R-module homomorphisms $r : Q \to E$ and $s : E \to S$ such that $rs = \mathrm{Id}_Q$. Let X and Y be R-modules, $\iota : X \to Y$ be a monomorphism of R-modules, and $g : X \to E$ be any R-module homomorphism. Then we have that $sg : X \to Q$, and since Q is injective, there exists an R-module homomorphism $h : Y \to Q$ such that $h\iota = sg$. Applying r to both sides of the equation we see that $rh\iota = (rh)\iota$ is equal to $rsg = (rs)g = g$. Thus rh is an R-module homomorphism with $(rh)\iota = g$, and since g and ι were chosen arbitrarily, this shows that E is an injective R-module. The argument is summarized by the following commutative diagram:

$$\begin{tikzcd}
E \arrow[r, "s"'] & Q \arrow[l, bend right, "r"'] \\
X \arrow[u, "g"] \arrow[r, hook, "\iota"'] & Y \arrow[u, dashed, "h"']
\end{tikzcd}$$

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