Group Theory

Homework Assignment II

Read all of the statements of the theorems in Chapter 5, and the proofs of the interesting theorems.

CHAPTER	**PROBLEMS**
5.1	**6 (I), 9 (II), 12, 13**
5.2	**1 (III), 6, 9, 10 (IV)**
5.3	**7**

ADDITIONAL PROBLEMS

1. (II) The n-th Engel word is the word

$$e_n(x,y) = [x, y, \ldots, y], \quad n\ y\text{'s}.$$

An n-Engel group is a group satisfying the law $e_n(x,y) = 1$.

(a) Show that a group is 2-Engel (satisfies $[x,y,y] = 1$)) iff every 1-generated normal subgroup is abelian.

(b) Show that the following laws are consequences of $[x,y,y] = 1$:

(i) $[x,y,z] = [y,z,x]$

(ii) $[x,y,z]^3 = 1$

(iii) $[x,y,z,t] = 1$

2. If $H \leq G$, show that $H \lhd G$ iff $[G,H] \subseteq H$.

3. (I) Show that if G is nilpotent and $a, b \in G$ have finite order, then the product ab also has finite order. (Hint: If $n = \mathrm{lcm}(|a|, |b|)$, then $a^n = b^n = 1$. Prove by induction on k that if G is nilpotent of class k, then $(ab)^{n^k} = 1$. For this it may help to replace G by the subgroup $\langle a, b\rangle$ and argue there. In this group, show that $\gamma_i(G)/\gamma_{i+1}(G)$ satisfies $x^n = 1$.)

4. (III) Extend the ascending central series of a group G,

$$1 = \zeta_0(G) \leq \zeta_1(G) \leq \ldots$$

transfinitely by taking unions at limit ordinals: $\zeta_\kappa(G) = \bigcup_{\lambda<\kappa} \zeta_\lambda(G)$. The largest term in this series is the *hypercenter* of G. G is *hypercentral* if it equals its hypercenter.

Show that if G is perfect (meaning that $[G, G] = G$), then the hypercenter of G is its center. (Hint: Use the Hall-Witt identity to show that if $H, K, L \lhd G$, then $[H, K, L] \leq [K, L, H][L, H, K]$. Then prove that $\zeta_2(G) = \zeta_1(G)$ by taking $(H, K, L) = (G, G, \zeta_2(G))$.)

5. Let G be a finite nonsolvable group. Let N be minimal among normal nonsolvable subgroups of G.

- **(a) Show that G has a normal subgroup, N_*, that is the largest for the property of being properly contained in N.**
- **(b) Show that every normal subgroup $K \lhd G$ satisfies either (i) $N \subseteq K$ or (ii) $K \subseteq (N_* : N)$, but no normal subgroup of G satisfies both (i) and (ii). ($(N_* : N) = \{g \in G | [g, N] \subseteq N_*\}$.)**
- **(c) Explain why G must be solvable if Norm(G) has no homomorphism onto the 2-element lattice.**

6. (IV) Find all finite solvable groups which have the property that Fit$(G) \cong \mathbb{Z}_5$.

7. Explain why, if G is a finite solvable group satisfying Fit$(G) \cong D_4$, it must be that $G \cong D_4$.

8. Explain why, if G is a finite solvable group satisfying Fit$(G) \cong Q_8$, it must be that $8 \leq |G| \leq 48$.

9. (I) An idempotent endomorphism ($\varphi^2 = \varphi$) is called a retraction; it is a proper retraction if $\varphi \neq$ *id*. Show that if φ is a proper retraction of a finite group G, then $\ker(\varphi) \not\leq \Phi(G)$.

10. Any ring has a Frattini subring (= the intersection of maximal subrings), and also a Frattini ideal (= the largest $I \lhd R$ such that $I \subseteq M$ for every maximal subring $M \leq R$).

- **(a) Show that these notions can be different. (Produce an example where the Frattini subring is not an ideal.)**
- **(b) State and prove the nongenerator properties (corresponding to Theorem 5.2.12 for the Frattini subgroup) for the Frattini subring of R and for the Frattini ideal of R.**

11. (II) Let G be a finite, 2-step nilpotent, p-group.

- **(a) Show that if p is odd, then G has an abelian group word, $w(x, y)$. That is, if $x \oplus y := w(x, y)$, then $\langle G; x \oplus y\rangle$ is an abelian group.**

(b) Is the same assertion true if p is even?
(c) Is the assertion true if G is a 3-step nilpotent p-group?

12. (III) Show that if P is a nonabelian group of order p^3, then any automorphism of the center of P extends to an automorphism of P.

13. For which $n \geq 0$ is there a finite group with exactly n nonnormal subgroups?

Collaboration Groups.

(I) DeLand, Jamesson, Orvis
(II) Willson, Doumont, Kuo
(III) Shearer, Wilson, Gensler
(IV) Lyness, Watson