Group Theory

Homework Assignment I

Read Sections 4.1 and 4.2.

PROBLEMS

(Notation: $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ = cyclic group of size n.)

1. (I) Show that a direct product of injective R-modules is injective.

2. (II) Show that if R is not left Noetherian, then some direct sum of injective R-modules will not be injective. (Hint: Suppose $I_0 \leq I_1 \leq I_2 \leq \cdots$ is an increasing chain of left ideals of R, and $I = \bigcup I_j$. For each j, let M_j be an injective extension of R/I_j. Combine the natural homomorphisms $\nu_j \colon I \to I/I_j \subseteq M_j$ into a single homomorphism $\nu \colon I \to \prod M_j$. Show that the image of ν lies in $\bigoplus M_j$. If the direct sum of the M_j's were injective, then one could extend ν to $\widehat{\nu} \colon R \to \bigoplus M_j$. Show that there is no such $\widehat{\nu}$.)

3. (III) Show that a retract of an injective R-module is injective.

4. (IV) Show that a quotient of an injective R-module need not be injective.

5. (I) Show that if A is an abelian group, then there is an abelian group D, unique up to isomorphism over A, such that

(i) $A \leq D$,

(ii) D is divisible, and

(iii) if $M \leq D$, $M \neq \{0\}$, then $A \cap M \neq \{0\}$.

D is called the divisible hull of A.

6. (II) If A is an abelian group, then the first Ulm subgroup of A is $A' = \bigcap_{n \in \mathbb{N}} nA$. (The second Ulm subgroup is $(A')'$, etc.)

Show that the first Ulm subgroup of a torsion-free group is divisible. (Hence $(A')' = A'$ when A is torsion-free.)

7. (III) Show that TFAE for nonzero abelian groups.

(i) D is divisible.

(ii) D has no maximal subgroups.

(iii) D has no finite nonzero quotient.

8. (IV) Show that is an abelian group A **contains elements** $a_0, a_1, a_2, \ldots$ **satisfying**

(i) $a_0 \neq 0$, **and**
(ii) $pa_{i+1} = a_i$ **holds for all** i,

then the subgroup of A **generated by these elements is isomorphic to** $\mathbb{Z}_{p^\infty}$**.**

9. (I) Show that if A **is an abelian group with the property that every nonzero quotient of** A **is isomorphic to** A, **then** $A \cong \mathbb{Z}_p$ **or** $A \cong \mathbb{Z}_{p^\infty}$**.**

10. (II) Suppose that A **is an infinite abelian group of cardinality** κ, **and every proper subgroup of** A **has cardinality** $< \kappa$. **Show that** $A \cong \mathbb{Z}_{p^\infty}$ **for some prime** p**.**

11. (III) Show that a torsion-free abelian group of cardinality κ **is embeddable in** $\oplus^\kappa \mathbb{Q}$**.**

Collaboration Groups.

(I) DeLand, Doumont, Gensler
(II) Jamesson, Kuo, Lyness
(III) Orvis, Shearer, Watson
(IV) Willson, Wilson