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Group Theory
Assignment 2

Exercise 5.2
1. If a nilpotent group has an element of prime order p, so does its center.

Proof. Assume G is a nilpotent group that contains an element $x \in G$ of prime order p.

Case 1: Suppose that $|G| < \infty$, say with unique prime factorization $|G| = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_t^{\alpha_t}$. Since G is a finite nilpotent group, then G is a direct product of its Sylow subgroups: $G \cong P_1 \times P_2 \times \cdots \times P_n$ where $|P_i| = p_i^{\alpha_i}$ for each $i \in \{1, 2, ..., t\}$. By Cauchy's Theorem, we know that G has elements of order p_i for each $i \in \{1, 2, ..., t\}$. Fix a prime p_j and consider $x \in G$ with $|x| = p_j$; then x is an element of the Sylow p-subgroup P_j. Additionally, $P_j \cap Z(G) \leq P_j$ so that $|P_j \cap Z(G)| = p_j^{\beta_j}$ for $0 \leq \beta_j \leq \alpha_j$ by Lagrange's Theorem. However, $P_j \trianglelefteq G$ implies that $P_j \cap Z(G) \neq \{1\}$ and hence $|P_j \cap Z(G)| = p_j^{\beta_j}$ for $1 \leq \beta_j \leq \alpha_j$. Since $P_j \cap Z(G)$ is a p_j-group, then it must have an element of order p_j by Cauchy's Theorem, and since $P_j \cap Z(G) \neq \{1\}$ is also a subgroup of $Z(G)$, then this implies $Z(G)$ must contain an element of order p_j.

Case 2: Suppose G has infinite order. Assume G has an element of prime order p. We will prove the center of G has an element of order p by induction. For the base case, suppose G is 2-step nilpotent. Let $a \in G \backslash \{1\}$ such that $a^p = 1$. If $a \in Z(G)$, then we are done. So suppose $a \notin Z(G)$. Then $\langle a \rangle \cap Z(G) = \{1\}$. Let $H = \langle a, Z(G) \rangle$. Note $a \notin Z(G)$, $a \in Z_2(G) = G$, and $a^p = 1 \in Z(G)$. So there exists $b \in G$ such that $[a, b] \in Z(G) \backslash \{1\}$, otherwise $a \in Z(G)$. Since the commutator $[a, b]$ commutes with $a, b \in G$ we have $[a, b]^n = [a^n, b]$ for $n \in \mathbb{Z}^+$, which we will prove by induction: Base case of $n = 1$ holds trivially. Suppose $[a, b]^{n-1} = [a^{n-1}, b]$. Observe

$$\begin{aligned}
[a, b]^n &= [a, b][a, b]^{n-1} \\
&= [a, b][a^{n-1}, b] \\
&= [a, b](a^{n-1} b (a^{n-1})^{-1} b^{-1}) \\
&= a^{n-1} [a, b] b (a^{n-1})^{-1} b^{-1} \\
&= a^{n-1} (a b a^{-1} b^{-1}) b (a^{n-1})^{-1} b^{-1} \\
&= a^n b a^{-1} (a^{n-1})^{-1} b^{-1} \\
&= a^n b (a^n)^{-1} b^{-1} \\
&= [a^n, b].
\end{aligned}$$

Letting $n = p$ we then have $[a, b]^p = [a^p, b] = [1, b] = 1$. Therefore, $[a, b]$ is an element of $Z(G) \backslash \{1\}$ of order p.

Now we will consider the case that G is n-step nilpotent. Note that $G/Z(G)$ is $n-1$-step nilpotent so by the induction hypothesis we may assume that the center of $G/Z(G)$ has an element of order p. That is, $Z_2(G)/Z(G) = Z(G/Z(G))$ has an element of order p. Let $a \in Z_2(G)/Z(G)$ be this element of order p. If $a \in Z(G)$ then we are done. So suppose not. Then $\bar{a} \in Z_2(G)/Z(G) \backslash \{\bar{1}\}$. So there exists $b \in G$ such that $[a, b] \in Z(G) \backslash \{1\}$, otherwise $a \in Z(G)$. Therefore, by the above argument in the base case of the induction argument on the nilpotence class of G we have $[a, b]^p = [a^p, b] = [1, b] = 1$ since the above proof showed this for a general $a, b \in G$ with $[a, b] \in Z(G) \backslash \{1\}$. Therefore, $[a, b]$ is an element of $Z(G) \backslash \{1\}$ of order p.

$\square$