Group Theory
Assignment 1

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11. Show that a torsion-free abelian group of cardinality κ is embeddable in $\oplus^{\kappa}\mathbb{Q}$.

Proof. Let A be a torsion-free abelian group of cardinality κ. Then A is isomorphic to a quotient of the free abelian group $\oplus^{\kappa}\mathbb{Z}$. We will express this as

$$A \cong \frac{\oplus^{\kappa}\mathbb{Z}}{R},$$

where R is a subgroup of the free abelian group over κ generators. Recall that $\mathbb{Z} \hookrightarrow \mathbb{Q}$ via inclusion is an embedding. By taking this embedding in each coordinate, we have that there is an embedding $\oplus^{\kappa}\mathbb{Z} \hookrightarrow \oplus^{\kappa}\mathbb{Q}$. Then we observe that $R \subseteq \oplus^{\kappa}\mathbb{Z} \subseteq \oplus^{\kappa}\mathbb{Q}$, so $Q := \oplus^{\kappa}\mathbb{Q}/R$ is well-defined and the previous embedding induces an embedding

$$A \hookrightarrow Q.$$

We will now show that Q is divisible. Let $(x_1, \ldots, x_m)R \in Q$, $(x_1, \ldots, x_m)R \neq 0$. We can represent elements of Q this way because only finitely many of the coordinates in a given element are nonzero. Let $n \in \mathbb{N}$. Then, $(x_1/n, \ldots, x_m/n)R \in Q$ and $n \cdot (x_1/n, \ldots, x_m/n)R = (x_1, \ldots, x_m)R$. Thus, Q is divisible.

Notice that Q is the quotient of an abelian group, and therefore abelian. So there exists a short exact sequence

$$\begin{tikzcd} & & A \arrow[d, hook] & & \\ 0 \arrow[r] & Q_T \arrow[r, "\iota"] & Q \arrow[r, "\pi"] & Q/Q_T \arrow[r] & 0 \end{tikzcd}$$

where Q_T is the torsion subgroup of Q. Since A is torsion-free, the image of the embedding $A \hookrightarrow Q$ is also torsion-free and therefore has trivial intersection with Q_T. Therefore, the composition $\phi : A \hookrightarrow Q \to Q/Q_T$ is an embedding. This is because if a is in the kernel of ϕ, then the image of a under the embedding $A \hookrightarrow Q$ must be in the kernel of π, but the kernel of π is Q_T. Therefore, a must be a torsion element in A, so $a = 0$ since A is torsion-free. Since quotients of divisible groups are divisible, we have that Q/Q_T is divisible and torsion-free. So, $Q/Q_T \cong \oplus^{\kappa_1}\mathbb{Q}$ for some cardinal κ_1. Hence we have shown that A is embeddable in $\oplus^{\kappa_1}\mathbb{Q}$.

Note that if $\kappa_1 \leq \kappa$, then we can embed $\oplus^{\kappa_1}\mathbb{Q}$ into $\oplus^{\kappa}\mathbb{Q}$, so we could embed A into $\oplus^{\kappa}\mathbb{Q}$. So instead, suppose that $\kappa_1 > \kappa$. Note that for each $a \in A$, $\phi(a)$ is nonzero in finitely many copies of $\mathbb{Q}$ (this follows from properties of the direct sum). So, the total number of nonzero copies of $\mathbb{Q}$ in $\oplus^{\kappa_1}\mathbb{Q}$ that intersect nontrivially with $\phi(A)$ is bounded by $\aleph_0 \cdot \kappa$. Note that since A is torsion-free, $|A| = \kappa$ is infinite (every finite group is torsion). Therefore, $\aleph_0 \cdot \kappa = \kappa$. Thus we have shown that only at most κ many of the copies of $\mathbb{Q}$ in $\oplus^{\kappa_1}\mathbb{Q}$ intersect nontrivially with $\phi(A)$. We will use the cardinal $\kappa_2 \leq \kappa$ to denote the number of copies of $\mathbb{Q}$ in $\oplus^{\kappa_1}\mathbb{Q}$ that intersect nontrivially with $\phi(A)$. Therefore, by composing ϕ with the projection onto $\oplus^{\kappa_2}\mathbb{Q}$ (up to permutation of coordinates), we still have an embedding. Now we are in the case where $\kappa_2 \leq \kappa$, so we can embed A into $\oplus^{\kappa}\mathbb{Q}$ by the previous argument. $\square$