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6. Let p be an odd prime. The two nonabelian groups of order p^3 have presentations

$$G_1 = \langle a, b \mid a^p = b^p = 1, [[a,b],b] = [[a,b],a] = 1\rangle$$

and

$$G_2 = \langle a, b \mid a^{p^2} = b^p = 1, [a,b] = a^p\rangle.$$

Let σ be a primitive p^2-th root of unity and let $\omega = \sigma^p$ be a primitive p-th root of unity. Consider the $p \times p$ matrices

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega & 0 & \cdots & 0 \\ 0 & 0 & \omega^2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{p-1} \end{bmatrix}, B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix}$$

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a) Show that $a \mapsto A$ and $b \mapsto B$ is an irreducible representation of G_1, and that $a \mapsto \sigma A$ and $b \mapsto B$ is an irreducible representation of G_2.

We show that A and B satisfy the relations given in the presentations for G_1 and G_2 respectively. This will show ρ is a homomorphism. First note that

$$A^p = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega^p & 0 & \cdots & 0 \\ 0 & 0 & \omega^{2p} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega^{(p-1)p} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Similarly, we have $(\sigma A)^{p^2} = I$. Also note

$$B^n = \begin{bmatrix} & & 1 & & \\ & & & \ddots & \\ & & & & 1 \\ 1 & & & & \\ & \ddots & & & \\ & & 1 & & \end{bmatrix}$$

where the 1's on each diagonal have been shifted down $n-1$ times. In particular, $B^p = 1$. We have

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \omega^{p-1} & 0 & \cdots & 0 \\ 0 & 0 & \omega^{p-2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \omega \end{bmatrix} \quad \text{and} \quad B^{-1} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & & \vdots \\ 0 & 0 & & \cdots & 1 \\ 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

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From this we can directly compute $[A, B] = \omega I$. In particular $[A, B]$ commutes with any matrix so the relations $[[A, B], B] = [[A, B], A] = 1$ are satisfied. This shows that $a \mapsto A$ and $b \mapsto B$ gives a homomorphism. Finally note that $[\sigma A, B] = (\sigma A)^{-1}B^{-1}\sigma AB = A^{-1}B^{-1}AB = [A, B] = \omega I = \sigma^p I = (\sigma A)^p$. This shows that $a \mapsto \sigma A$ and $b \mapsto B$ gives a homomorphism.

If the above maps were reducible they would be direct sums of representations of degree 1 because the above maps are of degree p. But if $\rho : G \to \mathbf{C}^*$ is a representation of degree 1, then in particular $\rho(a)$ commutes with $\rho(b)$. If ρ' is a direct sum of such representations, then $\rho'(a)$ commutes with $\rho'(b)$, which implies A commutes with B. This is a contradiction as we know from above that $[A, B] = [\sigma A, B] = \omega I \neq I$.

b) Show that G_1 and G_2 have the same character tables.

We will first determine the number of conjugacy classes of G_1 and G_2. Let G be any nonabelian group of order p^3 and note that G is an extraspecial p-group. This means we have $|Z(G)| = p$ and $Z(G) = G'$. We therefore have $|Z(G)| = p$ conjugacy classes of G with 1 element. Each other element g is in a larger conjugacy class. The conjugates of g are of the form $h^{-1}gh = g(g^{-1}h^{-1}gh)$. But $g^{-1}h^{-1}gh \in G' = Z(G)$. So every conjugate of g is the product of g with a central element of G and there are p central elements. So g has at most p conjugates. But The number of conjugates of g divides $|G| = p^3$, and g has more than 1 conjugate. Hence g has p conjugates. This shows that G has p conjugacy classes with 1 element and $\frac{|G|-|Z(G)|}{p} = p^2 - 1$ conjugacy classes with p elements each.

We will now compute degree 1 character values (these are all irreducible characters). In the case of G_1, note that any homomorphism $\rho : G_1 \to \mathbf{C}^*$ maps a and b to p-th roots of unity. And $\rho_{m,n} : G_1 \to \mathbf{C}^*$ given by $\rho_{m,n}(a) = \omega^m$ and $\rho_{m,n}(b) = \omega^n$ is a homomorphism and therefore each $\rho_{m,n}$ induces an irreducible character $\chi_{m,n} = \rho_{m,n}$. There are p choices for m and n which gives p^2 irreducible characters. Any element g of G_1 can be written as a product of powers of a and b and by the definition of $\rho_{m,n}$ we obtain the value $\rho_{m,n}(g) = \omega^{mk+nl}$ where k is the sum of exponents of a and l is the sum of the exponents of b. This determines the character table for the degree 1 irreducible characters. For G_2 note that the same $\rho_{m,n}$ gives a homomorphism into $\mathbf{C}^*$. Furthermore any homomorphism $\rho : G_2 \to \mathbf{C}^*$ sends b to a p-th root of unity and sends $[a, b]$ to the identity. But $[a, b] = a^p$ so ρ also sends a to a p-th root of unity. This shows that the character table for G_2 is the same as the character table for G_1 in the case of the degree 1 characters.

The sum of the squares of the degrees of the irreducible characters is p^3, which means that the remaining irreducible characters must have degree $< p^2$. Hence the remaining characters have degree p. There are $p - 1$ of them because there were p^2 characters of degree 1 and there are $p^2 + p - 1$ conjugacy classes. The maps given in part (a) yield irreducible characters of degree p. We already found that $[A, B] = \omega I$ so that if χ is the character corresponding to $a \mapsto A$ and $b \mapsto B$, we get $\chi([A, B]) = p\omega$. Similarly, we obtain $\chi([A, B]^n) = p\omega^n$. To compute the remaining part of the row of values of χ, we note that groups of order p^3 are Camina groups. By HW3 problem 4(b), this implies that χ takes the value 0 for every other element of the row. This gives us a row

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of the character table for G_1. Similarly we may obtain the same row for the character table of G_2 as $[\sigma A, B] = \omega I$ (as computed previously).

Finally, the remaining irreducible characters are obtained from the maps in part (a) by composing with automorphisms of G_1 and G_2. But in the case of groups of order p^3, every automorphism is induced by an automorphism of the center of the group (HW 2 problem 12). Since the center has order p, we know these automorphisms are given by $g \mapsto g^n$ where $1 \leq n \leq p-1$. Since the χ row of the character table is the same for G_1 and G_2, and the automorphisms of G_1 and G_2 are both given by $g \mapsto g^n$, we know the remaining rows of the character table are the same.

c) Show that the two groups can be distinguished by their determinant maps.

We first compute $\det A = 1 \cdot \omega \cdot \ldots \omega^{p-1} = \omega^{\frac{p(p-1)}{2}} = 1$ as p divides $\frac{p(p-1)}{2}$. By expanding along the first row of B we see that $\det B = 1$. We also have $\det A^n = 1^n = 1$ and $\det B^n = 1^n = 1$. Since irreducible representations of G_1 are given by composing the map from part (a) with automorphisms of G_1 and such autmorphisms are of the form $g \mapsto g^n$, we see that $\det \circ \rho = 1$ for each irreducible representation ρ of G_1 of degree p. But if $\rho' : G_2 \to \mathrm{Gl}(p, \mathbf{C})$ is the map from part (a) given by $a \mapsto \sigma A$ and $b \mapsto B$, we have $\det \circ \rho'(a) = \sigma^p = \omega \neq 1$. That is, the determinant of ρ' is not equal to the determinant of ρ for any irreducible representation ρ of G_1 of degree p.