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Group Theory
Assignment 2

(5.1.9) Show that the class of a nilpotent group cannot be bounded by a function of the derived length.

Proof. In order to show that the class of a nilpotent group cannot be bounded by a function of derived length, we will show that there exist groups of a fixed derived length with nilpotency classes that can be as large as we want. More precisely, we will show that the collection of Dihedral groups with order a power of 2 are a collection of groups all with derived length 2. But, they have ever increasing nilpotency class as we increase the size of the specific group under consideration.

Consider the Dihedral groups D_{2n} where n is a power of 2 say $n = 2^t$. These are all nilpotent since they are finite 2-groups (by Proposition 5.1.3, pg. 122 Robinson.) We know that D_{2n} has presentation

$$D_{2n} = \langle f, r \mid f^2 = 1 = r^n, f^{-1}rf = r^{-1}\rangle.$$

So, D_{2n} has a subgroup $\langle r\rangle$ of order n and this subgroup must be of index 2 which implies that it is normal in D_{2n}. Now, the quotient $D_{2n}/\langle r\rangle$ must be of order 2 and therefore is abelian. So, we have the abelian series

$$1 \lhd \langle r\rangle \lhd D_{2n}.$$

Since D_{2n} is not itself abelian, we have that the derived length is exactly 2.

Now, we show that the nilpotency class of D_{2n}, where n is a power of 2, is exactly n. We do this by showing that its lower central series has length n, since by Proposition 5.1.9 in Robinson this shows that the nilpotency class of D_{2n} is n. We know that with

$$\gamma_1 D_{2n} = D_{2n}$$

$$\gamma_2 D_{2n} = [D_{2n}, D_{2n}]$$

$$\gamma_{i+1} D_{2n} = [\gamma_i D_{2n}, D_{2n}]$$

then the lower central series is

$$\gamma_1 D_{2n} \geq \gamma_2 D_{2n} \geq \cdots .$$

We want to show that this series has length exactly n, and so we need to show that $\gamma_n D_{2n} = 1$ and $\gamma_i D_{2n} \neq 1$ for $i < n$. We do this by computing these groups explicitly.

We start with $[D_{2n}, D_{2n}]$. It is clear that

$$r^2 = f^{-1}r^{-1}fr$$

so $r^2 \in [D_{2n}, D_{2n}]$ and so $\langle r^2\rangle \leq [D_{2n}, D_{2n}]$. Since, $D_{2n}/\langle r^2\rangle$ has order 4, it must be abelian and so $\langle r^2\rangle \geq [D_{2n}, D_{2n}]$. Therefore $\langle r^2\rangle = [D_{2n}, D_{2n}] = \gamma_2 D_{2n}$.

Now, we consider $\gamma_i D_{2n} = [\gamma_{i-1} D_{2n}, D_{2n}]$ where $i > 2$. We use the induction hypothesis that $\gamma_{i-1} D_{2n} = \langle r^{2^{(i-1)}}\rangle$. First we can see that

$$[r^{2^{(i-1)}}, f] = r^{2^{(i-1)}} f r^{-2^{(i-1)}} f^{-1} = r^{2i}$$

so that $r^{2^i} \in \gamma_i D_{2n}$ and so $\langle r^{2^i} \rangle \leq \gamma_i D_{2n}$. We additionally know that

$$\gamma_i D_{2n} \leq \langle r^{2^{(i-1)}} \rangle = \gamma_{i-1} D_{2n}.$$

Now since $\langle r^{2^i} \rangle$ is a maximal subgroup of $\langle r^{2(i-1)} \rangle$, we must have either $\gamma_i D_{2n} = \langle r^{2^i} \rangle$ or $\gamma_i D_{2n} = \langle r^{2^{(i-1)}} \rangle$. Therefore, as long as we can show that $\gamma_i D_{2n} \neq \langle r^{2(i-1)} \rangle$ then we have $\langle r^{2^i} \rangle = \gamma_i D_{2n}$.

Suppose by way of contradiction that $\gamma_i D_{2n} = \langle r^{2^{(i-1)}} \rangle$. Then we would have

$$\gamma_i D_{2n} = \gamma_{i-1} D_{2n},$$

and so for any $j > i$ we would have

$$\gamma_j D_{2n} = [\gamma_{j-1} D_{2n}, D_{2n}] = \cdots = [\gamma_{i-1} D_{2n}, D_{2n}] = \gamma_i D_2 n.$$

But this would show that the lower central series does not terminate. This would contradict the fact that D_{2n} is nilpotent, which we know as a consequence of it being a 2-group (since n is a power of 2.) Therefore, $\langle r^{2^i} \rangle = \gamma_i D_{2n}$.

Then this shows that the lower central series of D_{2n} is

$$D_{2n} \geq \langle r^2 \rangle \geq \langle r^4 \rangle \geq \cdots \geq \langle r^{2^t} \rangle = \langle r^n \rangle = 1$$

and therefore the nilpotency class of D_{2n} is n as desired. This further shows that the class of a nilpotent group cannot be bounded by a function of the derived length. $\square$