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Problem 10. Suppose that A is an infinite abelian group of cardinality κ, and every proper subgroup of A has cardinality $< \kappa$. Show that $A \cong \mathbb{Z}_{p^\infty}$ for some prime p.

Proof. Note that A is indecomposable, because if $A = G \oplus H$ where neither G nor H are trivial, then G, H both embed as proper subgroups of A. However, $|A| = \max\{|G|, |H|\}$, so one of G or H must have cardinality κ.

We now show that A is divisible. Suppose otherwise, then there exists $n \in \mathbb{N} \setminus \{0\}$ and $a \in A$ such that $nx = a$ has no solution in A. That is, nA is a proper subgroup of A. If $nA = 0$, then $A \cong \oplus_{i \in I} C_i$ where each C_i is a cyclic group of order $\leq n$, but this contradicts that A is indecomposable. Thus, if $\lambda_n : A \to A$ is the homomorphism taking $a \mapsto na$, then $\ker(\lambda_n)$ is also a proper subgroup of A. However, A is the disjoint union of cosets of $\ker \lambda_n$, and there is precisely one coset for each element of nA. That is, A is in bijection (just as sets) with $\ker \lambda_n \times nA$. Thus $|A| = |\ker(\lambda_n) \times nA| = \max\{|\ker(\lambda_n)|, |nA|\}$. Hence, A contains a proper subgroup of the same cardinality.

It follows that A must be an indecomposable divisible group, so A must be isomorphic to $\mathbb{Q}$ or $\mathbb{Z}_{p^\infty}$ for some prime p. Now, $\mathbb{Q}$ contains nontrivial subgroups of the same order, so it follows that $A \cong \mathbb{Z}_{p^\infty}$ for some p. $\square$