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5.1.6 Suppose that G is a nilpotent group which is not abelian and let $g \in G$. Show that the nilpotent class of $\langle g, G' \rangle$ is smaller than that of G. Deduce that G can be expressed as a product of normal subgroups of smaller class.

Claim 1. *If G not abelian and $G/\zeta_k(G)$ is cyclic, then $G = \zeta_k(G)$.*

Proof of claim 1. Suppose $G/\zeta_k(G)$ is cyclic. By the third isomorphism theorem,

$$G/\zeta_k(G) \cong \left(G/\zeta_{k-1}(G)\right) / \left(\zeta_k(G)/\zeta_{k-1}(G)\right).$$

By definition of the upper central series,

$$\zeta_k(G)/\zeta_{k-1}(G) = Z(G/\zeta_{k-1}).$$

Recall that if a group modulo its center is cyclic, then the group is abelian, so $G/\zeta_{k-1}(G)$ is abelian, and in particular,

$$Z(G/\zeta_{k-1}(G)) = G/\zeta_{k-1}(G) = \zeta_k(G)/\zeta_{k-1}(G).$$

Since $\zeta_k(G)$ is a subgroup of G and $G/\zeta_{k-1}(G) = \zeta_k(G)/\zeta_{k-1}(G)$, we have $G = \zeta_k(G)$. $\square$

Claim 2. *If $H = \langle g, G' \rangle$, then $\zeta_n(G) \cap H \subseteq \zeta_n(H)$.*

Proof of claim 2. Note that $\zeta_0(G) \cap H = \{1\} \subseteq \zeta_0(H)$. Now by induction, suppose that $\zeta_n(G) \cap H \subseteq \zeta_n(H)$. We need to show that $\zeta_{n+1}(G) \cap H \subseteq \zeta_{n+1}(H)$. Let $a \in \zeta_{n+1}(G) \cap H$ and let $b \in H$. Since $a \in \zeta_{n+1}(G)$, $a\zeta_n(G) \in Z(G/\zeta_n(G))$. Therefore

$$a\zeta_n(G)(b\zeta_n(G)) = b\zeta_n(G)(a\zeta_n(G))$$
$$(ab)\zeta_n(G) = (ba)\zeta_n(G)$$

In particular, $ab = ba\gamma$ for some $\gamma \in \zeta_n(G)$. Now note that since a and b are both in H, we must therefore have $\gamma \in H$, and hence $\gamma \in H \cap \zeta_n(G)$. Then by the inductive hypothesis, $\gamma \in \zeta_n(H)$. Then we have

$$(ab)\zeta_n(H) = (ba)\zeta_n(H),$$

so $a\zeta_n(H) \in Z(H/\zeta_n(H))$. Therefore $a \in \zeta_{n+1}(H)$. $\square$

Proof of Problem 5.1.6. Suppose G is not abelian and of nilpotent class k, and let $H = \langle g, G' \rangle$. We first want to show that $H/\zeta_{k-1}(H)$ is cyclic, so we can apply claim 1 to $H/\zeta_{k-1}(H)$. Recall that G' is normal in G, so elements of H can be written as $g^m h$ for some $h \in G'$ and $m \in \mathbb{Z}$. Then the elements of $H/\zeta_{k-1}(H)$ must be the cosets $\overline{g^m h}$. Note that Theorem 5.1.9(ii) in Robinson implies that $G' \leq \zeta_{k-1}(G)$ ($k \geq 2$ since G is not abelian). Then by claim 2,

$$G' \subseteq \zeta_{k-1}(G) \cap H \subseteq \zeta_{k-1}(H).$$

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Therefore $\overline{h} = 1$, so $\overline{g^m h} = \overline{g^m}$. We can therefore generate the cosets of $H/\zeta_{k-1}(H)$ by $\overline{g}$, so $H/\zeta_{k-1}(H)$ is cyclic. Now by claim 1, we have $H = \zeta_{k-1}(H)$, so H must be of nilpotent class smaller than that of G.

Finally, we want to show that G can be expressed as a product of normal subgroups of smaller class. Note that since G/G' is abelian, all subgroups in G/G' are normal. Then a subgroup of G/G' must correspond to a normal subgroup of G by the Fourth Isomorphism Theorem. In particular, for $g \notin G'$, $\langle g, G' \rangle$ is normal in G, and by what we proved above, has smaller nilpotent class. If we take the product of all such groups $\langle g, G' \rangle$ with $g \notin G'$, the product must be equal to G. Hence G can be expressed as a product of normal subgroups of smaller class. $\square$