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4. Let G be a finite group and for $g \in G$, denote the conjugacy class of g in G as

$$g^G = \{h \in G \mid h = x^{-1}gx \text{ for some } x \in G\}.$$

(a) Show that the conjugacy classes outside of G' are contained in cosets: $g^G \subseteq gG'$ for $g \notin G'$.

(b) Show that if conjugacy classes outside of G' are equal to cosets, i.e. $g^G = gG'$ for all $g \notin G'$, then every $\chi \in \mathrm{Irr}(G)$ with $\chi(1) > 1$ vanishes off of G'.

Proof. First, recall the column orthogonality relations included below.

> Column Orthogonality Relations. Let G be a finite group and denote $\mathrm{Irr}(G)$ as the collection of non-equivalent irreducible characters of G over $\mathbb{C}$. Then,
>
> $$\sum_{\chi \in \mathrm{Irr}(G)} \chi(g)\overline{\chi(h)} = \begin{cases} |C_G(g)| & \text{if } h \in g^G \\ 0 & \text{otherwise} \end{cases}$$
>
> where $\overline{\chi(h)}$ is the complex conjugate of $\chi(h)$ and $C_G(g)$ is the centralizer of g in G.

Let G be a finite group. Recall that the commutator subgroup G' is normal in G and the corresponding quotient group G/G' is abelian. Throughout, suppose g is an element of G such that $g \notin G'$.

(a) Let $h \in g^G$; then $h = x^{-1}gx$ for some $x \in G$. Furthermore,

$$\begin{aligned} hG' &= x^{-1}gxG' \\ &= (x^{-1}G')(gG')(xG') \\ &= (gG')(x^{-1}G')(xG') && (G/G' \text{ is abelian}) \\ &= (gG')(x^{-1}xG') \\ &= (gG')(eG') \\ &= (ge)G' \\ &= gG'. \end{aligned}$$

Thus $h \in gG'$ and $g^G \subseteq gG'$.

(b) Assume that the conjugacy classes of $g \notin G'$ are equal to cosets, meaning that $hG' = gG'$ if and only if $h \in g^G$. By the Orbit-Stabilizer Theorem, $|C_G(g)| = \frac{|G|}{|g^G|}$ and because $g^G = gG'$ by assumption, this becomes $|C_G(g)| = \frac{|G|}{|gG'|}$. Furthermore, $|gG'| = |G'|$ implies

$$|C_G(g)| = |G : G'|.$$

Let $\mathrm{Irr}(G) = \{\chi_i\}_{i=1}^{s}$ be the complete set of pairwise non-equivalent irreducible characters of G, meaning that the corresponding irreducible representations of G are pairwise non-isomorphic. For later convenience, let the subset $\{\chi_i\}_{i=1}^{t} \subseteq \mathrm{Irr}(G)$ correspond to the linear characters of G and notice that this implies $t = |G : G'|$. By the column orthogonality relations on the $\{g\}$-column of the character table for G,

$$|C_G(g)| = \sum_{i=1}^{s} \chi_i(g)\overline{\chi_i(g)} = \sum_{i=1}^{t} \chi_i(g)\overline{\chi_i(g)} + \sum_{\{i \ \mid \ \chi_i(1)>1\}} \chi_i(g)\overline{\chi_i(g)}. \tag{1}$$

Now consider the group G/G'. Since G/G' is abelian, all of its irreducible characters are linear and hence $t = |\mathrm{Irr}(G/G')|$ follows from the character table relation $|G/G'| = \sum_{\tilde{\chi}\in\mathrm{Irr}(G/G')}[\tilde{\chi}(1G')]^2$. Denote the complete set of pairwise non-equivalent irreducible characters of G/G' by $\mathrm{Irr}(G/G') = \{\tilde{\chi}_i\}_{i=1}^{t}$. Notice also that the fact G/G' is an abelian group implies $|C_{G/G'}(gG')| = |G/G'| = |G : G'|$. So, using the column orthogonality relation on the $\{gG'\}$-column of the character table for G/G',

$$|G : G'| = \sum_{i=1}^{t} \tilde{\chi}_i(gG')\overline{\tilde{\chi}_i(gG')}. \tag{2}$$

As established above, $|C_G(g)| = |G : G'|$; hence Equations (1) and (2) are equal and

$$\sum_{\{i \ \mid \ \chi_i(1)>1\}} \chi_i(g)\overline{\chi_i(g)} = \sum_{i=1}^{t} \tilde{\chi}_i(gG')\overline{\tilde{\chi}_i(gG')} - \sum_{i=1}^{t} \chi_i(g)\overline{\chi_i(g)}. \tag{3}$$

We claim that the right side of this equation is zero. To show this, let $\tilde{\rho}_i : G/G' \to \mathbb{C}^\times$ denote the irreducible representation of G/G' that corresponds to the irreducible character $\tilde{\chi}_i'$. If $\pi : G \to G/G'$ is the projection $x \mapsto xG'$, then we get a representation $\rho_i = \tilde{\rho}_i \circ \pi : G \to \mathbb{C}^\times$ of G via inflation. Even further, we can conclude this G-representation ρ_i is linear and irreducible because it is the inflation of an irreducible G/G'-representation. Therefore, the trace of ρ_i corresponds to some linear element of $\mathrm{Irr}(G)$, say χ_i. For the element $g \notin G'$,

$$\rho_i(g) = \tilde{\rho}_i \circ \pi(g) = \tilde{\rho}_i(gG')$$

and thus $\chi_i(g) = \tilde{\chi}_i(gG')$ for each $i \in \{1, 2, ..., t\}$. So Equation (3) becomes

$$\sum_{\{i \ \mid \ \chi_i(1)>1\}} \chi_i(g)\overline{\chi_i(g)} = 0.$$

Since each $\chi_i(g)\overline{\chi_i(g)}$ in the above summand is a non-negative real number and the entire sum equals zero, we conclude that $\chi_i(g) = 0$ for each $\chi_i \in \mathrm{Irr}(G)$ such that $\chi_i(1) > 1$. In other words, each non-linear irreducible character of G vanishes off of the commutator G'.

□