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9. An idempotent endomorphism ($\varphi^2 = \varphi$) is called a retraction; it is a proper retraction if $\varphi \neq \text{id}$. Show that if φ is a proper retraction of a finite group G, then $\ker(\varphi) \not\leq \Phi(G)$.

Proof. We will first show that $G = (\ker \varphi)\varphi(G)$. To show this, note that for $g \in G$, we have $g = g(\varphi(g))^{-1}\varphi(g)$. But

$$\varphi(g(\varphi(g))^{-1}) = \varphi(g)(\varphi^2(g))^{-1} = \varphi(g)(\varphi(g))^{-1} = 1$$

implies that $g(\varphi(g))^{-1} \in \ker \varphi$. Also $\varphi(g) \in \varphi(G)$, which shows $g \in (\ker \varphi)\varphi(G)$.

Now we claim $\ker \varphi \neq \{1\}$. If we had $\ker \varphi = 1$, then φ would be injective and hence surjective as G is finite, which means $\varphi \in \text{Aut}(G)$. Again, since G is finite, we know that the order of φ is finite, say n. Then $\varphi^n = id$, but by hypothesis, we obtain $\varphi^n = \varphi$ so that $\varphi = id$, a contradiction. This also shows that $\varphi(G)$ is a proper subgroup of G.

Now because G is finite, there is a maximal proper subgroup M of G containing $\varphi(G)$. Such a subgroup cannot contain $\ker \varphi$ by the above (otherwise we would have $M = G$, a contradiction). Therefore $\Phi(G) \leq M$ cannot contain $\ker \varphi$ as desired. $\square$