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Exercise 2.5
How many inequivalent irreducible $\mathbb{C}$-representations does S_n have? In the case of S_4, find the degrees of these representations.

Proof. First, since distinct irreducible representations are in one-to-one correspondence with conjugacy classes finding how many inequivalent irreducible $\mathbb{C}$-representations there are of S_n is the same as finding how many conjugacy classes S_n has. Further, we know the conjugacy classes of S_n correspond to the different cycle types, so we just have to find how many distinct cycle types there are of size n. Now, the number of cycle types correspond to the number of integer partitions of n. Thus, the number of distinct irreducible $\mathcal{C}$-representations of S_n is exactly the number of integer partitions there are of n.

To answer the second part of the problems, note that there are 5 integer partitions of 4, so we need to find the degrees of the 5 distinct irreducible representations. Further, we know that $|S_4| = 24$ is equal to the sum of the squares of the different degrees. Next, we already know two linear representations, namely the trivial representation and the sign representation. This reduces the problem to finding three numbers whose squares sum to 22. That is, we need a, b, and c such that $a^2 + b^2 + c^2 = 22$. One possibility is $a = 2$, $b = 3$, and $c = 3$. We will show this is the only option. To that end, note that none of these variables can be 4 as that would require us to write 6 as the sum of two squares which is impossible. Next, note if we choose two of the variables to be 2, then it would necessitate $14 = 22 - 4 - 4$ being a square, which isn't the case. Thus, the only possibility is $a = 2$, $b = 3$, and $c = 3$. Thus the degrees of the five irreducible representations for S_4 are 1, 1, 2, 3, and 3. $\square$