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9. Show that if A is an abelian group with the property that every nonzero quotient of A is isomorphic to A, then $A \simeq \mathbb{Z}_p$ or $A \simeq \mathbb{Z}_{p^\infty}$.

Proof. First consider the case where A is finite. If $N \neq 0$ is a subgroup of A (so $|N| > 1$), then $|A/N| = \frac{|A|}{|N|} < |A|$. Therefore the condition that every nonzero quotient of A is isomorphic to A implies that the only subgroups of the finite group A are 0 and A (so A is simple). Now Cauchy's Theorem implies that if p is a prime dividing $|A|$, then A has an element of order p. As this element generates a subgroup that is not the trivial subgroup, it must generate all of A. Therefore $A \cong \mathbb{Z}_p$.

Now consider the case where A is infinite. To show that $A \cong \mathbb{Z}_{p^\infty}$, we will show A has the same properties as $\mathbb{Z}_{p^\infty}$: that it has a unique subgroup of order p contained in all other nonzero subgroups, that it has unique subgroups of order $\mathbb{Z}_{p^k}$, and that it is the union of these subgroups.

First we will show that A has a unique nonzero subgroup that is contained in all of the nonzero subgroups of A. Let $a \in A$ and let N be a subgroup of A that is maximal such that $a \notin N$. Note that N is normal in A since A is abelian, and A/N is a nonzero quotient of A since $a + N \in A/N$ is nonzero. Consider the subgroup $M = \langle a + N \rangle \subset A/N$. This must be a subgroup of every nonzero subgroup of A/N since otherwise there would be a nonzero subgroup $M' \subset A/N$ such that $a + N \not\subset M'$. Then by the fourth isomorphism theorem, M' corresponds to a subgroup of A strictly containing N that does not contain a, which contradicts N being maximal with this property. Therefore M is in every nonzero subgroup of A/N. Now observe that M has no nontrivial proper subgroups, so M is cyclic since for $x \in M$ with $x \neq 0$, $\langle x \rangle = M$. Also observe that M must be finite since infinite cyclic groups have nontrivial proper subgroups. Therefore by the finite case, $M \cong \mathbb{Z}_p$. Since A/N is isomorphic to A and A/N has a unique subgroup of order p which is in every nonzero subgroup, A must also have a unique subgroup of order p which is in every nonzero subgroup. Call this subgroup M_1.

We will now show that A has unique subgroups of order $\mathbb{Z}_{p^k}$ for all finite k. Consider A/M_1. Since this is isomorphic to A, it has a subgroup of order $\mathbb{Z}_q$. Then pulling this subgroup back along the natural map from A to A/M_1, we get a subgroup M_2 of A of order $\mathbb{Z}_{pq}$. Since A had a unique minimal subgroup contained in all subgroups, we must have $p = q$, so $M_2 \cong \mathbb{Z}_{p^2}$. M_2 also contains all elements of order p^2. Continuing in this manner, if we consider A/M_k where $M_k \cong \mathbb{Z}_{p^k}$, it has a subgroup of order p, so pulling it back gives us a subgroup of order $\mathbb{Z}_{p^{k+1}}$.

Finally, we will show that A is the union of these M_i's by showing it is torsion and the union of the M_i's contains all torsion elements of A. To see that A is torsion, let $x \in A$ and consider $\langle x \rangle$. Since $M_1 \subset \langle x \rangle$, $mx \in M_1 \cong \mathbb{Z}_p$ for some $m \in \mathbb{Z}$. Since p annihilates every element of $\mathbb{Z}_p$, $pmx = 0$, so x had to be torsion. However, any torsion element has to be in a subgroup of order $\mathbb{Z}_{p^k}$ for some k, so A is in the union of all subgroups of order $\mathbb{Z}_{p^k}$. Therefore $A \cong \mathbb{Z}_{p^\infty}$. $\square$