Jennifer Gensler
Calum Shearer
Breeann Wilson

4. Extend the ascending central series of a group G,

$$1 = \zeta_0(G) \leq \zeta_1(G) \leq \dots$$

transfinitely by taking unions at limit ordinals: $\zeta_\kappa(G) = \bigcup_{\lambda<\kappa} \zeta_\lambda(G)$. The largest term in this series is the *hypercenter* of G. G is *hypercentral* if it equals its hypercenter.
Show that if G is perfect (meaning that $[G,G] = G$), then the hypercenter of G is its center. (Hint: Use the Hall-Witt identity to show that if $H, K, L \lhd G$, then $[H,K,L] \leq [K,L,H][L,H,K]$. Then prove that $\zeta_2(G) = \zeta_1(G)$ by taking $(H,K,L) = (G,G,\zeta_2(G))$.)

Proof. First, assume that H, K, and L are normal subgroups of G. Then $[K,L,H] \trianglelefteq G$ and $[L,H,K] \trianglelefteq G$, which implies that the product $N = [K,L,H][L,H,K]$ is also a normal subgroup of G. Recall that, in general, a commutator subgroup $[X,Y,Z]$ is generated by elements of the form $[x,y,z]$, or equivalently, it is generated by elements of the form $[x,y^{-1},z]$ where $x \in X$, $y \in Y$, and $z \in Z$. Thus $[K,L,H]$ is generated by elements of the form $[k,\ell^{-1},h]$ where $k \in K$, $\ell \in L$, and $h \in H$. Since $N \trianglelefteq G$ and $[k,\ell^{-1},h] \in N$, then $\ell^{-1}[k,\ell^{-1},h]\ell = [k,\ell^{-1},h]^\ell$ is an element of N. Similarly, $[\ell,h^{-1},k] \in N$ implies that $h^{-1}[\ell,h^{-1},k]h = [\ell,h^{-1},k]^h$ is an element of N. So by the closure of the group N, $[k,\ell^{-1},h]^\ell[\ell,h^{-1},k]^h \in N$. By the Hall-Witt identity,

$$[h,k^{-1},\ell]^k[k,\ell^{-1},h]^\ell[\ell,h^{-1},k]^h = 1$$
$$[h,k^{-1},\ell]^k = \left([k,\ell^{-1},h]^\ell[\ell,h^{-1},k]^h\right)^{-1}.$$

Hence $[h,k^{-1},\ell]^k$ is an element of N because the element on the right side of the above equation is in N. Conjugating by k, we get that

$$k\left([h,k^{-1},\ell]^k\right)k^{-1} = k\left(k^{-1}[h,k^{-1},\ell]k\right)k^{-1} = [h,k^{-1},\ell].$$

Since $N \trianglelefteq G$, this shows $[h,k^{-1},\ell] \in N$ for any $h \in H$, $k \in K$, and $\ell \in L$. But $[H,K,L]$ is generated by elements of the form $[h,k^{-1},\ell]$; hence $[H,K,L]$ is a subgroup of $N = [K,L,H][L,H,K]$ whenever $H,K,L \trianglelefteq G$.

By the logic above, $[G,G,\zeta_2(G)] = [[G,G],\zeta_2(G)] \leq [G,\zeta_2(G),G][\zeta_2(G),G,G]$ because G and $\zeta_2(G)$ are both normal subgroups of G. However, G is a perfect group by assumption, so this statement becomes

$$[G,\zeta_2(G)] \leq [[G,\zeta_2(G)],G][[\zeta_2(G),G],G].$$

The definition of the i'th center as $\zeta_{i+1}/\zeta_i = Z(G/\zeta_i)$ tells us that $[G,\zeta_{i+1}(G)] \leq \zeta_i(G)$ for all i, and similarly $[\zeta_{i+1}(G),G] \leq \zeta_i(G)$ by anti-symmetry. In particular, $[G,\zeta_2(G)]$ and $[\zeta_2(G),G]$ are both subgroups of $\zeta_1(G) = Z(G)$. Applying this fact to $[G,\zeta_2(G)] \leq [[G,\zeta_2(G)],G][[\zeta_2(G),G],G]$ and using the monotonicity of the commutator, we get that

$$[G,\zeta_2(G)] \leq [\zeta_1(G),G][\zeta_1(G),G] \leq \{1\}\{1\} = 1$$

This tells us that elements of $\zeta_2(G)$ commute with all elements of G, i.e. $\zeta_2(G) \leq Z(G) = \zeta_1(G)$. But we also have that $\zeta_1(G) \leq \zeta_2(G)$ by construction of the upper central series, and

Jennifer Gensler
Calum Shearer
Breeann Wilson

hence we know there is equality $\zeta_1(G) = \zeta_2(G)$. Since $\zeta_2 = \zeta_1$, it follows by induction that $\zeta_\lambda = \zeta_1$ for all λ; so the upper central series is simply given by $1 \leq Z(G) \leq Z(G) \leq \cdots$ where all terms except the first are $Z(G)$. Thus the hypercenter of G is $1 \cup Z(G) = Z(G)$.

$\square$