Rebecca DeLand
Bob Kuo
Matthew Watson

Problem 8.1.1
Prove that Maschke's Theorem does not hold if:

(i) The characteristic of the field divides the order of the group; or

(ii) The group is infinite.

To disprove Maschke's Theorem in each case, we will find counterexamples of $k[G]$ modules that do not have a complemented submodule lattice. We start with the case when G is not finite.

Example 1. Suppose $G = \mathbb{Z}$. If $A = k[G] \cong k[T, T^{-1}]$, then an A-module M is a k-vectorspace along with an invertible linear transformation $T : M \to M$. That is, since $\mathbb{Z} \cong \{T^n : n \in \mathbb{Z}\}$ (where the right hand side is a multiplicative group), then the action of T^n on M is determined by the action of T. For M to have an A-module structure, it suffices for T to act as an invertible linear transformation.

Specifically, $V = k \oplus k$, and $T : V \to V$ be defined by $(x, y) \mapsto (x, x + y)$. Then the A-submodules of V are k-subspaces that are invariant under the G-action of V. Since the action is determined by T, then A-submodules are just T invariant subspaces. Furthermore, V is two-dimensional, so its only nontrivial, proper subspaces are subspaces generated by an eigenvector of T. With respect to the standard basis, T is represented by the matrix

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

which has characteristic polynomial $(x - 1)^2$. Thus, 1 is the only eigenvalue of T, and eigenvectors are $[x\ y]$ such that $T(x, y) = (x, x + y) = (x, y)$. This forces $x = 0$ because $x + y = y$, and leaves y free, so we see that the only nontrivial, proper submodule of V is $W = \{(0, y) : y \in k\}$. However, a complement of W is necessarily 1-dimensional, so it follows that V does not have a complemented A-submodule lattice. Note that this argument works for any field k.

A counterexample for when G is finite, but the characteristic of k divides $|G|$ only requires a slight modification of Example 1.

Example 2. We will construct a similar example to Example 1, except replacing k with $\mathbb{Z}/p\mathbb{Z}$ and G with C_p for some prime p. Let g be a generator of G then an $A = k[G]$-module M is the data of a k-vectorspace M and an action of g on M as a linear transformation T that satisfies $T^p = I$. Since

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^p = \begin{bmatrix} 1 & 0 \\ p & 1 \end{bmatrix} \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \pmod p$$

then as with the previous example, letting g act via the transformation $T(x, y) = (x, x + y)$ gives $V = k \oplus k$ the structure of an A-module. Now, the same argument as before still applies, so $\{(0, y) : y \in k\}$ is a submodule of V with no complemented A-submodule.