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3. Show that if $G$ is nilpotent and $a, b \in G$ have finite order, then the product $ab$ also has finite order. (Hint: If $n = \mathrm{lcm}(|a|, |b|)$, then $a^n = b^n = 1$. Prove by induction on $k$ that if $G$ is nilpotent of class $k$, then $(ab)^{n^k} = 1$. For the induction argument, first consider $k = 1$. Then assume the induction claim is true in $G/\gamma_k(G)$ and deduce it for $G$.)

*Proof.* First suppose that $G$ is nilpotent of class $k = 1$. Then $[G, G] = \{1\}$, and in particular, $G$ is abelian. Therefore $(ab)^n = a^n b^n = 1$.

Now we will proceed by induction. First, note that since a subgroup of a nilpotent group is nilpotent, and all powers of $ab$ are contained in $\langle a, b\rangle$, we may assume that $G = \langle a, b\rangle$ without loss of generality. Let the lower central series for $G$ be given by

$$\gamma_1(G) = G, \gamma_2(G) = [G, G], \ldots, \gamma_k(G) = [\gamma_{k-1}(G), G], \gamma_{k+1}(G) = \{1\},$$

so that in particular $G$ is $k$-step nilpotent. We show by induction on $k$ that $\gamma_r(G)/\gamma_{r+1}(G)$ is a torsion group for all $1 \leq r \leq k-1$. If $r = 1$, then $\gamma_1(G)/\gamma_2(G) = G/[G, G]$, which is abelian. By the argument above, the torsion elements of an abelian group form a subgroup, so the result holds in this case. Next, suppose that the result holds for all $1 \leq r$, and consider $\gamma_{r+1}(G)/\gamma_{r+2}(G)$. Since $\gamma_{r+1}(G) = [\gamma_r(G), G]$, we have that $\gamma_{r+1}(G)/\gamma_{r+2}(G)$ is generated by elements of the form $\overline{[x, g]}$ with $x \in \gamma_r(G)$ and $g \in G$, where the bar represents the image of $[x, g]$ under the quotient map. By assumption we have that $\gamma_r(G)/\gamma_{r+1}(G)$ is torsion, so that there exists an $m$ such that $x^m \in \gamma_{r+1}(G)$. Consider $\overline{[x, g]}^m$. We show that $\gamma_{r+1}(G)/\gamma_{r+2}(G) \subset Z(G/\gamma_{r+2}(G))$. To this end, let $x \in \gamma_{r+1}(G)$ and $g \in G$, with $\overline{x}$ and $\overline{g}$ representing the cosets $x\gamma_{r+2}(G)$ and $g\gamma_{r+2}(G)$ as before. Then $[\overline{x}, \overline{g}] = \overline{[x, g]} = 1$ since $[x, g] \in [\gamma_{r+1}(G), G] = \gamma_{r+2}(G)$. Thus for any $\overline{g} \in G/\gamma_{r+2}(G)$ we have that $[\overline{x}, \overline{g}] = 1$, so $\overline{x}$ commutes with $\overline{g}$. Since $\overline{x}$ is a generic element of $\gamma_{r+1}(G)/\gamma_{r+2}(G)$, we have that $\gamma_{r+1}(G)/\gamma_{r+2}(G) \subset Z(G/\gamma_{r+2}(G))$. Using this fact, we have that $\overline{[x, g]} \in Z(G/\gamma_{r+2}(G))$, because $[x, g] \in \gamma_{r+1}(G)$ since $x \in \gamma_r(G)$. Thus $\overline{[x^m, g]} = \overline{[x, g]}^{\overline{x}}\overline{[x^{m-1}, g]} = \overline{[x, g][x^{m-1}, g]}$ where the bar represents the coset of $\gamma_{r+2}(G)$. Applying this argument repeatedly on $\overline{[x^{m-1}, g]}$ gives that $\overline{[x^m, g]} = \overline{[x, g]}^m$. Since $x^m \in \gamma_{r+1}(G)$, we have that $[x^m, g] \in [\gamma_{r+1}(G), G] = \gamma_{r+2}(G)$, and so $\overline{[x^m, g]} = 1$. We have that $[\gamma_{r+1}(G), \gamma_{r+1}(G)] \subset [\gamma_{r+1}(G), G] = \gamma_{r+2}(G)$, so $\gamma_{r+1}(G)/\gamma_{r+2}(G)$ is abelian. Thus $\gamma_{r+1}(G)/\gamma_{r+2}(G)$ is a finitely generated abelian group with torsion generators, and is therefore torsion as desired. This completes the inductive proof, so we have that $\gamma_r(G)/\gamma_{r+1}(G)$ is torsion for all $1 \leq r \leq k-1$. We now show that this implies that $ab$ is torsion in $G$. Continuing the inductive argument on the nilpotence class of $G$, we assume that the result holds for any group of nilpotence class less than $k$. The nilpotence class of $G/\gamma_{k-1}(G)$ is $k-2$, so we have by the inductive hypothesis that since $\overline{a}$ and $\overline{b}$ are torsion in this group, so is $\overline{ab}$. Let $m$ be such that $\overline{ab}^m = 1$. Then $(ab)^m \in \gamma_{k-1}(G)$. By the previous result, we have that $\gamma_{k-1}(G)/\gamma_k(G)$ is torsion, so there exists $k$ such that $(\overline{(ab)^m})^k = 1$. Thus $((ab)^m)^k \in \gamma_k(G) = \{1\}$, so $(ab)^{mk} = 1$, and $ab$ is torsion in $G$. ∎