6. Find all finite solvable groups G which have the property that $\mathrm{Fit}(G) \cong \mathbb{Z}_5$.

Proof. Let G be a group such that $\mathrm{Fit}(G) = F \cong \mathbb{Z}_5$. Then since G is finite and solvable $C_G(F) = Z(F)$. Since F is normal G acts on F by conjugation, with an arbitrary element g acting by the automorphism: $\gamma_g(x) = g^{-1}xg$. This induces a homomorphism:

$$\begin{aligned} \Gamma : G &\to \mathrm{Aut}(F) \\ g &\mapsto \gamma_g \end{aligned}$$

Note that $\mathrm{Ker}(\Gamma) = C_G(F) = Z(F)$. Applying all this to our situation where $F \cong \mathbb{Z}_5$ we see that $Z(F) \cong \mathbb{Z}_5$ and $\mathrm{Aut}(F) \cong \mathbb{Z}_4$. By the first isomorphism theorem:

$$\frac{G}{\mathbb{Z}_5} \leq \mathrm{Aut}(F)$$

Since $\mathrm{Aut}(F) \cong \mathbb{Z}_4$ we see that $G/\mathbb{Z}_5$ is isomorphic to $\mathbb{Z}_4$ or $\mathbb{Z}_2$ or $\{0\}$. We will consider these cases separately.

1. Suppose $\mathrm{Aut}(F) \cong \mathbb{Z}_4$. Then:
$$\frac{G}{\mathbb{Z}_5} \cong \mathbb{Z}_4$$
This implies that G contains an element of order 4, hence $G \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_4$. The possibilities for G are as follows:

   (a) $G = \mathbb{Z}_{20}$
   (b) $G = \langle x, y \mid y^{-1}xy = x^2, x^5 = 1 = y^4 \rangle$
   (c) $G = \langle x, y \mid y^{-1}xy = x^3, x^5 = 1 = y^4 \rangle$
   (d) $G = \langle x, y \mid y^{-1}xy = x^{-1}, x^5 = 1 = y^4 \rangle$

   (a) $\mathrm{Fit}(G) = \mathbb{Z}_{20}$ which doesn't work.
   (b) $\mathbb{Z}_5 \cong \langle x \rangle \lhd G$ which implies that $\langle x \rangle \leq \mathrm{Fit}(G)$. If $\mathrm{Fit}(G) \neq x$ then $\mathrm{Fit}(G) = \langle x, y^k \rangle$ where k is either 1 or 2. This implies that $\langle x, y^2 \rangle \leq \mathrm{Fit}(G)$, where $\langle x, y^2 \rangle \cong D_{10}$. This is a contradiction since D_{10} is not nilpotent. Hence $\mathrm{Fit}(G) = \langle x \rangle \cong \mathbb{Z}_5$. This works.
   (c) This group is isomorphic to the group in (b) by the isomorphism sending $x \mapsto x$ and $y \mapsto y^{-1}$.
   (d) $y^{-2}xy^2 = x$, which implies that x and y^2 commute. Hence $\langle x, y^2 \rangle \cong Z_{10}$. This is nilpotent, and is a normal subgroup since it has index 2. Hence $\langle x, y^2 \rangle \leq \mathrm{Fit}(G)$ which doesn't work.

2. Suppose $\mathrm{Aut}(F) \cong \mathbb{Z}_2$. Then $G \cong \mathbb{Z}_5 \rtimes \mathbb{Z}_2$. The possibilities for G are as follows:

   (a) $G = \mathbb{Z}_{10}$

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(b) $G = \langle r, s \mid rs = sr^{-1}, r^5 = 1 = s^2 \rangle = D_{10}$

(a) $\text{Fit}(G) = \mathbb{Z}_{10}$ which doesn't work.

(b) $\langle r \rangle$ is a nilpotent normal subgroup, so $\langle r \rangle \leq \text{Fit}(G) \leq G$. This implies that $\text{Fit}(G)$ is either $\langle r \rangle$ or G since $\langle r \rangle$ is an index 2 subgroup of G. Since $G = D_{10}$ is not nilpotent, it must be that $\text{Fit}(G) = \langle r \rangle \cong \mathbb{Z}_5$. This works.

3. Suppose $\text{Aut}(F) = \{0\}$. Then $G \cong \mathbb{Z}_5$, so $\text{Fit}(G) \cong \mathbb{Z}_5$. This works.

Thus the only possibilities for G are $\langle x, y \mid y^{-1}xy = x^2, x^5 = 1 = y^4 \rangle$, D_{10} and $\mathbb{Z}_5$.

$\square$