The Group Determinant

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Example

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. After replacing the elements of the table with commuting variables, we obtain $\begin{bmatrix} t_1 & t_g \\ t_g & t_1 \end{bmatrix}$*, whose determinant is* $\Theta(G) = t_1^2 - t_g^2 = (t_1 + t_g)(t_1 - t_g)$.

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This factors over $\mathbb{R}$ *as* $(t_1 + t_g + t_{g^2})(t_1^2 + t_g^2 + t_{g^2}^2 - 2(t_1 t_g + t_1 t_{g^2} + t_g t_{g^2}))$,

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If $K = \{1, a, b, c\}$, *then*

$$\begin{aligned}\Theta(K) &= (t_1^4 + t_a^4 + t_b^4 + t_c^4) \\ &\quad -2(t_1^2 t_a^2 + t_1^2 t_b^2 + t_1^2 t_c^2 + t_a^2 t_b^2 + t_a^2 t_c^2 + t_b^2 t_c^2) + 8 t_1 t_a t_b t_c\end{aligned}$$

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(This is a silly way to prove that $\mathbb{Z}_4 \ncong \mathbb{Z}_2 \times \mathbb{Z}_2$.)

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	1	g	g^2
χ_1	1	1	1
χ_2	1	ω	ω^2
χ_3	1	ω^2	ω

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Using the characters for the three element group G, we can write the factorization of $\Theta(G)$ over $\mathbb{C}$ as

$$(t_1+t_g+t_{g^2})(t_1+\omega t_g+\omega^2 t_{g^2})(t_1+\omega^2 t_g+\omega t_{g^2}) = \prod_{i=1}^{3}(\chi_i(1)t_1+\chi_i(g)t_g+\chi_i(g^2)t_{g^2}).$$

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This is an instance of a general phenomenon:

Theorem

Let G be a finite abelian group with dual group $\widehat{G}$. The factorization of the group determinant is $\Theta(G) = \prod_{\chi\in\widehat{G}} P_\chi$ where $P_\chi = \left(\sum_{g\in G}\chi(g)t_g\right)$.

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$$\begin{bmatrix} t_{g_1 g_1^{-1}} & t_{g_1 g_2^{-1}} & \cdots & t_{g_1 g_n^{-1}} \\ t_{g_2 g_1^{-1}} & t_{g_2 g_2^{-1}} & & t_{g_2 g_n^{-1}} \\ \vdots & & \ddots & \vdots \\ t_{g_n g_1^{-1}} & t_{g_n g_2^{-1}} & \cdots & t_{g_n g_n^{-1}} \end{bmatrix} \cdot \begin{bmatrix} \chi(g_1^{-1}) \\ \chi(g_2^{-1}) \\ \vdots \\ \chi(g_n^{-1}) \end{bmatrix} = P_\chi \cdot \begin{bmatrix} \chi(g_1^{-1}) \\ \chi(g_2^{-1}) \\ \vdots \\ \chi(g_n^{-1}) \end{bmatrix}$$

Nonabelian groups

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Example

If $D_3 = \{1, r, r^2, f, rf, r^2f\}$ *is the dihedral group, then* $\Theta(G)$ *is the product of the homogeneous factors* $(t_1 + t_r + t_{r^2} + t_f + t_{rf} + t_{r^2f})$,
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Definition

A *character* of a finite group G is a function $\chi\colon G \to \mathbb{C}$ that can be factored as $G \xrightarrow{\rho} \mathrm{GL}(d,\mathbb{C}) = M_d(\mathbb{C})^* \xrightarrow{\mathtt{tr}} \mathbb{C}$ where ρ is a group homomorphism and `tr` is the trace map.

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(ii) $$\chi^{(r)}(g_1, g_2, \dots, g_r) = \chi(g_1)\chi^{(r-1)}(g_2, g_3, \dots, g_r) - \chi^{(r-1)}(g_1 \cdot g_2, g_3, \dots, g_r) - \chi^{(r-1)}(g_2, g_1 \cdot g_3, \dots, g_r) - \cdots - \chi^{(r-1)}(g_2, g_3, \dots, g_1 \cdot g_r).$$

The extension of Dedekind's Theorem

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Let G be a finite group, and let $\mathcal{X} = \{\chi_1, \dots, \chi_m\}$ be a complete set of irreducible characters of G. Then $|\mathcal{X}|$ equals the class number of G, and the complete factorization of the group determinant is $\Theta(G) = \prod_{\chi \in \mathcal{X}} P_\chi$ where $P_\chi = \frac{1}{d!}\left(\sum_{\overline{g} \in G^d} \chi^{(d)}(\overline{g}) t_{\overline{g}}\right)^d$ if the degree of χ is d. Here if $\overline{g} = (g_{i_1}, g_{i_2}, \dots, g_{i_d})$, then $t_{\overline{g}} = t_{g_{i_1}} t_{g_{i_2}} \cdots t_{g_{i_d}}$.

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It is clear that G determines both $\Theta(G)$ and the set of k-characters. The theorem shows that the k-characters of G determine $\Theta(G)$. It has since been shown (by Formanek and Sibley (1991)) that $\Theta(G)$ determines G up to isomorphism. This was improved (by Hoehnke and Johnson (1992)) to show that the 1-, 2-, and 3-characters of G determine G up to isomorphism.