Examples, Properties, Applications

G	1 / 1	k_2 / g_2	$\cdots$ / $\cdots$	k_r / g_r
χ_1	1	1	$\cdots$	1
χ_2	d_2	$\chi_2(g_2)$	$\cdots$	$\chi_2(g_r)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
χ_r	d_r	$\chi_r(g_2)$	$\cdots$	$\chi_r(g_r)$

Each d_i divides $|G| = 1^2 + d_2^2 + \cdots + d_r^2$
Each k_j divides $|G| = 1 + k_2 + \cdots + k_r$

Character tables of abelian groups

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$\mathbb{Z}_2$	1 0	1 1
ξ_1	1	1
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$\mathbb{Z}_3$	1 0	1 1	1 2
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χ_2	1	ω	ω^2
χ_2^2	1	ω^2	ω

Character tables of abelian groups

$\mathbb{Z}_2$	$\begin{matrix}1\\0\end{matrix}$	$\begin{matrix}1\\1\end{matrix}$
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χ_1	1	1	1	1
χ_2	1	i	-1	$-i$
χ_2^2	1	-1	1	-1
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$\mathbb{Z}_2 \times \mathbb{Z}_2$	$\begin{matrix}1\\(0,0)\end{matrix}$	$\begin{matrix}1\\(0,1)\end{matrix}$	$\begin{matrix}1\\(1,0)\end{matrix}$	$\begin{matrix}1\\(1,1)\end{matrix}$
$\xi_1(x)\xi_1(y)$	1	1	1	1
$\xi_2(x)\xi_1(y)$	1	1	-1	-1
$\xi_1(x)\xi_2(y)$	1	-1	1	-1
$\xi_2(x)\xi_2(y)$	1	-1	-1	1

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χ_4	1	1	-1	-1	1
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The entries in the last row can be determined by:
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The entries in the last row can be determined by:
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χ_4	1	1	-1	-1	1
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The table for either of $D_4(D_8)$ or Q_8 is:

	1	1	2	2	2
G	1	g_2	g_3	g_4	g_5
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χ_3	1	1	-1	1	-1
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The groups D_4 and Q_8 can be distinguished by the fact that $\det(\chi_5^{Q_8}) = \chi_1 \neq \det(\chi_5^{D_4})$.

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$\det(\chi_5^{Q_8}) = \chi_1 \neq \det(\chi_5^{D_4})$.

$$Q_8:\ i \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, j \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \quad D_4:\ f \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, r \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

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S_4	1 1	3 (1 2)(3 4)	6 (1 2)	6 (1 2 3 4)	8 (1 2 3)
χ_1	1	1	1	1	1
χ_2	1	1	-1	-1	1
χ_3	2	2	0	0	-1
χ_4	3	-1	1	-1	0
χ_5	3	-1	-1	1	0

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A_5	1 1	15 (1 2)(3 4)	20 (1 2 3)	12 (1 2 3 4 5)	12 (1 2 3 5 4)
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	1	15	20	12	12
A_5	1	(1 2)(3 4)	(1 2 3)	(1 2 3 4 5)	(1 2 3 5 4)
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π	5	1	2	0	0

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$\pi - \chi_1$	4	0	1	-1	-1

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A_5 acts 2-transitively by conjugation on its six Sylow 5-subgroups with character

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A_5 may be realized as the rotation group of the dodecahedron, yielding an irrep of degree 3.

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A_5	1 1	15 (1 2)(3 4)	20 (1 2 3)	12 (1 2 3 4 5)	12 (1 2 3 5 4)
θ	$1+2\cos(0)$	$1+2\cos(\pi)$	$1+2\cos(2\pi/3)$	$1+2\cos(2\pi/5)$	$1+2\cos(4\pi/5)$
θ	3	-1	0	ϕ	$-\phi^{-1}$

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A_5	1 1	15 $(1\ 2)(3\ 4)$	20 $(1\ 2\ 3)$	12 $(1\ 2\ 3\ 4\ 5)$	12 $(1\ 2\ 3\ 5\ 4)$
θ	$1 + 2\cos(0)$	$1 + 2\cos(\pi)$	$1 + 2\cos(2\pi/3)$	$1 + 2\cos(2\pi/5)$	$1 + 2\cos(4\pi/5)$
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If α is the automorphism $\sigma \mapsto (4\ 5)^{-1}\sigma(4\ 5)$ of A_5, then $\theta \circ \alpha$ is irreducible:

A_5	1 1	15 $(1\ 2)(3\ 4)$	20 $(1\ 2\ 3)$	12 $(1\ 2\ 3\ 4\ 5)$	12 $(1\ 2\ 3\ 5\ 4)$
$\theta \circ \alpha$	3	-1	0	$-\phi^{-1}$	ϕ

A_5

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χ_3	3	-1	0	$-\phi^{-1}$	ϕ
χ_4	4	0	1	-1	-1
χ_5	5	1	-1	0	0

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Silly applications

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If $|G| = p$, p prime, then G is abelian.

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The first column of the character table for G consists of at most p integers $d_1, \ldots, d_r$ such that (i) $d_1 = 1$, (ii) $d_j | p$ for all j, and (iii) $d_1^2 + \cdots + d_r^2 = p$.

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The first column of the character table for G consists of at most p integers $d_1, \ldots, d_r$ such that (i) $d_1 = 1$, (ii) $d_j | p$ for all j, and (iii) $d_1^2 + \cdots + d_r^2 = p$. It must be that $d_1 = \cdots = d_r = 1$, so G is abelian. □

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The first column of the character table for G consists of at most p integers $d_1, \ldots, d_r$ such that (i) $d_1 = 1$, (ii) $d_j | p$ for all j, and (iii) $d_1^2 + \cdots + d_r^2 = p$. It must be that $d_1 = \cdots = d_r = 1$, so G is abelian. □

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Any group of order < 6 is abelian.

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Same idea. The smallest number n that is a sum of squares $d_1^2 + \cdots + d_r^2$ where $d_1 = 1$, some $d_j > 1$, and all d_j divide n is 6. □

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Each summand on the left is an algebraic integer, but $-1/p$ is not. □

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A theorem about simple groups

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χ_1	1	1	1	1	1
χ_2	3	-1	0	ϕ	$-\phi^{-1}$
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There are zeros at the centers of the colored crosses.

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What we are seeing here is important:

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What we are seeing here is important: a character is vanishing on the nonidentity elements of a subgroup, $Z(P)$.

Recognizing multiples of the regular representation

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Lemma

Let $\rho\colon H \to GL(n, \mathbb{C})$ be a (not necessarily irreducible) representation of a finite group, and let ψ be the character afforded by ρ. If ψ vanishes on $H \setminus \{1\}$, then $|H|$ divides $\psi(1)$.

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Proof.

Compare χ_1 and ψ:

H	1 / 1	k_2 / a_2	$\cdots$ / $\cdots$	k_r / a_r
χ_1	1	1	$\cdots$	1
ψ	$\psi(1)$	0	$\cdots$	0

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Proof.

Compare χ_1 and ψ:

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$\langle \chi_1, \psi \rangle = \frac{\psi(1)}{|H|}$ must be an integer.

Recognizing multiples of the regular representation

Lemma

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$\langle \chi_1, \psi \rangle = \frac{\psi(1)}{|H|}$ must be an integer. Now use that representations are determined by their characters. $\square$

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To be proved: If G is simple and there is an irreducible representation $\rho\colon G \to \mathrm{GL}(p, \mathbb{C})$ (affording ψ) then a Sylow p-subgroup $P \leq G$ has order p.

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The converse is false.

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The converse is false. $\mathrm{PSL}_2(q)$ has Sylow 3-subgroups of size 3 for infinitely many prime powers q, but only $\mathrm{PSL}_2(5)$ and $\mathrm{PSL}_2(7)$ have irreps of degree 3.