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Alternating implies antisymmetric. Converse holds if $\text{char}(\mathbb{F}) \neq 2$.

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so the value of $B(u, v)$ is determined by $[u]_{\mathcal{B}}, [v]_{\mathcal{B}}$, and the values $B(e_i, e_j)$. In fact, if $[B]_{\mathcal{B}} = [B(e_i, e_j)] = M$ then

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$$B(u, v) = [u]_{\mathcal{B}}^t [B]_{\mathcal{B}} [v]_{\mathcal{B}} = [u]^t M [v].$$

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Call matrices $M$ and $N$ **congruent** if there is an invertible matrix $T$ such that $M = T^tNT$. We wish to classify possible matrices for a symplectic form $B(x, y)$, up to congruence.

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8. $B(x, y)|_{H_1^\perp}$ is a symplectic form on $H_1^\perp$.

# The proof

1. Let $e_1$ be any nonzero vector.
2. $e_1^\perp$ has codimension 1, so choose $f_1$ such that $B(e_1, f_1) \neq 0$. May assume, after scaling $f_1$, that $B(e_1, f_1) = 1$.
3. $e_1^\perp \neq f_1^\perp$, since $f_1 \in f_1^\perp - e_1^\perp$. (Similarly, $e_1 \in e_1^\perp - f_1^\perp$.)
4. Let $H_1 = \operatorname{span}(e_1, f_1)$.
5. $B(x,y)|_{H_1}$ has matrix $S$ relative to $\{e_1, f_1\}$.
6. $H_1^\perp = e_1^\perp \cap f_1^\perp$ has codimension 2.
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9. Matrix of $B$ on $H_1 \oplus H_1^\perp$ has form $\left[\begin{array}{c|c} S & 0 \\ \hline 0 & * \end{array}\right]$.