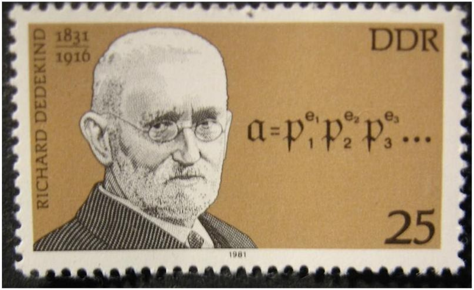
# Modules


RICHARD DEDEKIND
1831
1916
DDR
$\mathfrak{a} = \mathfrak{p}_1^{e_1} \mathfrak{p}_2^{e_2} \mathfrak{p}_3^{e_3} \dots$
25
1981

# Modules, I

# Modules, I

Richard Dedekind introduced the name “module” into mathematics.

# Modules, I

Richard Dedekind introduced the name “module” into mathematics. (From “modulus of a congruence $a \equiv b \pmod{p}$”

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$"

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$"

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.**

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.** ($R$ a ring.)

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle M; +, -, 0\rangle$ is an abelian group and

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle M; +, -, 0\rangle$ is an abelian group and $\rho : R \to \mathrm{End}_{\mathbb{Z}}(M) : r \mapsto r(x)$ is a ring homomorphism.

# Modules, I

Richard Dedekind introduced the name “module” into mathematics. (From “modulus of a congruence $a \equiv b \pmod{p}$” to “modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$” to “additive subgroup of $\mathbb{C}$” to “module”.)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle M; +, -, 0\rangle$ is an abelian group and $\rho : R \to \mathrm{End}_{\mathbb{Z}}(M) : r \mapsto r(x)$ is a ring homomorphism.

**Df.**

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle M; +, -, 0\rangle$ is an abelian group and $\rho : R \to \mathrm{End}_{\mathbb{Z}}(M) : r \mapsto r(x)$ is a ring homomorphism.

**Df.** ($R$ a $k$-algebra

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle M; +, -, 0\rangle$ is an abelian group and $\rho : R \to \mathrm{End}_{\mathbb{Z}}(M) : r \mapsto r(x)$ is a ring homomorphism.

**Df.** ($R$ a $k$-algebra defined by $k \to R : \alpha \mapsto \alpha \cdot 1$.)

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle M; +, -, 0\rangle$ is an abelian group and $\rho : R \to \mathrm{End}_{\mathbb{Z}}(M) : r \mapsto r(x)$ is a ring homomorphism.

**Df.** ($R$ a $k$-algebra defined by $k \to R : \alpha \mapsto \alpha \cdot 1$.) An $R$-module is a structure $\langle V; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle M; +, -, 0\rangle$ is an abelian group and $\rho : R \to \mathrm{End}_{\mathbb{Z}}(M) : r \mapsto r(x)$ is a ring homomorphism.

**Df.** ($R$ a $k$-algebra defined by $k \to R : \alpha \mapsto \alpha \cdot 1$.) An $R$-module is a structure $\langle V; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle V; +, -, 0, \{\alpha(x) \mid \alpha \in k\}\rangle$ is $k$-vector space and

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle M; +, -, 0\rangle$ is an abelian group and $\rho : R \to \mathrm{End}_{\mathbb{Z}}(M) : r \mapsto r(x)$ is a ring homomorphism.

**Df.** ($R$ a $k$-algebra defined by $k \to R : \alpha \mapsto \alpha \cdot 1$.) An $R$-module is a structure $\langle V; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle V; +, -, 0, \{\alpha(x) \mid \alpha \in k\}\rangle$ is $k$-vector space and $\rho : R \to \mathrm{End}_k(V) : r \mapsto r(x)$ is a $k$-algebra homomorphism.

# Modules, I

Richard Dedekind introduced the name “module” into mathematics. (From “modulus of a congruence $a \equiv b \pmod{p}$” to “modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$” to “additive subgroup of $\mathbb{C}$” to “module”.)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle M; +, -, 0\rangle$ is an abelian group and $\rho : R \to \mathrm{End}_{\mathbb{Z}}(M) : r \mapsto r(x)$ is a ring homomorphism.

**Df.** ($R$ a $k$-algebra defined by $k \to R : \alpha \mapsto \alpha \cdot 1$.) An $R$-module is a structure $\langle V; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle V; +, -, 0, \{\alpha(x) \mid \alpha \in k\}\rangle$ is $k$-vector space and $\rho : R \to \mathrm{End}_k(V) : r \mapsto r(x)$ is a $k$-algebra homomorphism.

The class of all $R$-modules is equationally definable,

# Modules, I

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle M; +, -, 0\rangle$ is an abelian group and $\rho : R \to \mathrm{End}_{\mathbb{Z}}(M) : r \mapsto r(x)$ is a ring homomorphism.

**Df.** ($R$ a $k$-algebra defined by $k \to R : \alpha \mapsto \alpha \cdot 1$.) An $R$-module is a structure $\langle V; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle V; +, -, 0, \{\alpha(x) \mid \alpha \in k\}\rangle$ is $k$-vector space and $\rho : R \to \mathrm{End}_k(V) : r \mapsto r(x)$ is a $k$-algebra homomorphism.

The class of all $R$-modules is equationally definable, hence forms (the object class of) a complete and cocomplete category.

Richard Dedekind introduced the name "module" into mathematics. (From "modulus of a congruence $a \equiv b \pmod{p}$" to "modulus of a congruence $a \equiv b \pmod{\mathfrak{p}}$" to "additive subgroup of $\mathbb{C}$" to "module".)

**Df.** ($R$ a ring.) An $R$-module is a structure $\langle M; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle M; +, -, 0\rangle$ is an abelian group and $\rho : R \to \mathrm{End}_{\mathbb{Z}}(M) : r \mapsto r(x)$ is a ring homomorphism.

**Df.** ($R$ a $k$-algebra defined by $k \to R : \alpha \mapsto \alpha \cdot 1$.) An $R$-module is a structure $\langle V; +, -, 0, \{r(x) \mid r \in R\}\rangle$ such that $\langle V; +, -, 0, \{\alpha(x) \mid \alpha \in k\}\rangle$ is $k$-vector space and $\rho : R \to \mathrm{End}_k(V) : r \mapsto r(x)$ is a $k$-algebra homomorphism.

The class of all $R$-modules is equationally definable, hence forms (the object class of) a complete and cocomplete category. The study of this category is the representation theory of $R$.

# Simplest examples

# Simplest examples

1. $(R = \mathbb{Z})$ :

# Simplest examples

1. $(R = \mathbb{Z})$ :

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$

2. $(R = k,\ \text{a field})$ :

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$

2. $(R = k,\ \text{a field})$ :

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$
2. $(R = k, \text{ a field})$ : A $k$-module is a $k$-vector space $V$.

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$
2. $(R = k, \text{ a field})$ : A $k$-module is a $k$-vector space $V$. So the representations of a field $k$ are classified by dimension.

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$
2. $(R = k, \text{ a field})$ : A $k$-module is a $k$-vector space $V$. So the representations of a field $k$ are classified by dimension.
3. $(R = M_2(\mathbb{R}))$ :

## Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$
2. $(R = k,\ \text{a field})$ : A $k$-module is a $k$-vector space $V$. So the representations of a field $k$ are classified by dimension.
3. $(R = M_2(\mathbb{R}))$ :

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$
2. $(R = k, \text{ a field})$ : A $k$-module is a $k$-vector space $V$. So the representations of a field $k$ are classified by dimension.
3. $(R = M_2(\mathbb{R}))$ : $\mathbb{R}^2$ is a simple $M_2(\mathbb{R})$-module.

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$
2. $(R = k, \text{ a field})$ : A $k$-module is a $k$-vector space $V$. So the representations of a field $k$ are classified by dimension.
3. $(R = M_2(\mathbb{R}))$ : $\mathbb{R}^2$ is a simple $M_2(\mathbb{R})$-module. Every $M_2(\mathbb{R})$-module is a direct sum of a uniquely determined number of copies of this simple module.

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$
2. $(R = k, \text{ a field})$ : A $k$-module is a $k$-vector space $V$. So the representations of a field $k$ are classified by dimension.
3. $(R = M_2(\mathbb{R}))$ : $\mathbb{R}^2$ is a simple $M_2(\mathbb{R})$-module. Every $M_2(\mathbb{R})$-module is a direct sum of a uniquely determined number of copies of this simple module. The representations of $M_2(\mathbb{R})$ are also classified by a single cardinal parameter.

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$
2. $(R = k, \text{ a field})$ : A $k$-module is a $k$-vector space $V$. So the representations of a field $k$ are classified by dimension.
3. $(R = M_2(\mathbb{R}))$ : $\mathbb{R}^2$ is a simple $M_2(\mathbb{R})$-module. Every $M_2(\mathbb{R})$-module is a direct sum of a uniquely determined number of copies of this simple module. The representations of $M_2(\mathbb{R})$ are also classified by a single cardinal parameter.
4. $(R = R)$ :

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$

2. $(R = k,\ \text{a field})$ : A $k$-module is a $k$-vector space $V$. So the representations of a field $k$ are classified by dimension.
3. $(R = M_2(\mathbb{R}))$ : $\mathbb{R}^2$ is a simple $M_2(\mathbb{R})$-module. Every $M_2(\mathbb{R})$-module is a direct sum of a uniquely determined number of copies of this simple module. The representations of $M_2(\mathbb{R})$ are also classified by a single cardinal parameter.
4. $(R = R)$ :

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$
2. $(R = k, \text{ a field})$ : A $k$-module is a $k$-vector space $V$. So the representations of a field $k$ are classified by dimension.
3. $(R = M_2(\mathbb{R}))$ : $\mathbb{R}^2$ is a simple $M_2(\mathbb{R})$-module. Every $M_2(\mathbb{R})$-module is a direct sum of a uniquely determined number of copies of this simple module. The representations of $M_2(\mathbb{R})$ are also classified by a single cardinal parameter.
4. $(R = R)$ : $R$ is an $R$-module.

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$
2. $(R = k,\ \text{a field})$ : A $k$-module is a $k$-vector space $V$. So the representations of a field $k$ are classified by dimension.
3. $(R = M_2(\mathbb{R}))$ : $\mathbb{R}^2$ is a simple $M_2(\mathbb{R})$-module. Every $M_2(\mathbb{R})$-module is a direct sum of a uniquely determined number of copies of this simple module. The representations of $M_2(\mathbb{R})$ are also classified by a single cardinal parameter.
4. $(R = R)$ : $R$ is an $R$-module. (Write ${}_RR$.)

# Simplest examples

1. $(R = \mathbb{Z})$ : A $\mathbb{Z}$-module is an abelian group $M$ equipped with scalar multiplications
$$n(x) = \underbrace{x + \cdots + x}_{n \text{ times}}, \qquad n \in \mathbb{Z}$$
2. $(R = k,\ \text{a field})$ : A $k$-module is a $k$-vector space $V$. So the representations of a field $k$ are classified by dimension.
3. $(R = M_2(\mathbb{R}))$ : $\mathbb{R}^2$ is a simple $M_2(\mathbb{R})$-module. Every $M_2(\mathbb{R})$-module is a direct sum of a uniquely determined number of copies of this simple module. The representations of $M_2(\mathbb{R})$ are also classified by a single cardinal parameter.
4. $(R = R)$ : $R$ is an $R$-module. (Write ${}_RR$.) The submodules of ${}_RR$ are the (left) ideals of $R$.

# Topics to discuss

# Topics to discuss

1. Products, coproducts.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.
3. Tensor product.

# Topics to discuss

1. Products, coproducts.
    (a) Universal properties.
    (b) Constructions of products and coproducts.
2. Free objects.
    (a) Construction.
    (b) Presentations.
3. Tensor product.
    (a) Representable functors.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.
3. Tensor product.
   (a) Representable functors.
   (b) Coalgebras.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.
3. Tensor product.
   (a) Representable functors.
   (b) Coalgebras.
   (c) Presentation of tensor product.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.
3. Tensor product.
   (a) Representable functors.
   (b) Coalgebras.
   (c) Presentation of tensor product.
4. Restriction and extension of scalars.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.
3. Tensor product.
   (a) Representable functors.
   (b) Coalgebras.
   (c) Presentation of tensor product.
4. Restriction and extension of scalars.
5. Exact sequences.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.
3. Tensor product.
   (a) Representable functors.
   (b) Coalgebras.
   (c) Presentation of tensor product.
4. Restriction and extension of scalars.
5. Exact sequences.
   (a) Exact functors.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.
3. Tensor product.
   (a) Representable functors.
   (b) Coalgebras.
   (c) Presentation of tensor product.
4. Restriction and extension of scalars.
5. Exact sequences.
   (a) Exact functors.
   (b) Projectives and injectives.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.
3. Tensor product.
   (a) Representable functors.
   (b) Coalgebras.
   (c) Presentation of tensor product.
4. Restriction and extension of scalars.
5. Exact sequences.
   (a) Exact functors.
   (b) Projectives and injectives.
   (c) Flatness.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.
3. Tensor product.
   (a) Representable functors.
   (b) Coalgebras.
   (c) Presentation of tensor product.
4. Restriction and extension of scalars.
5. Exact sequences.
   (a) Exact functors.
   (b) Projectives and injectives.
   (c) Flatness.
6. Localization.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.
3. Tensor product.
   (a) Representable functors.
   (b) Coalgebras.
   (c) Presentation of tensor product.
4. Restriction and extension of scalars.
5. Exact sequences.
   (a) Exact functors.
   (b) Projectives and injectives.
   (c) Flatness.
6. Localization.

# Topics to discuss

1. Products, coproducts.
   (a) Universal properties.
   (b) Constructions of products and coproducts.
2. Free objects.
   (a) Construction.
   (b) Presentations.
3. Tensor product.
   (a) Representable functors.
   (b) Coalgebras.
   (c) Presentation of tensor product.
4. Restriction and extension of scalars.
5. Exact sequences.
   (a) Exact functors.
   (b) Projectives and injectives.
   (c) Flatness.
6. Localization.

Please read A-M, Chapters 2 & 3.

# Universal morphisms

# Universal morphisms

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$,

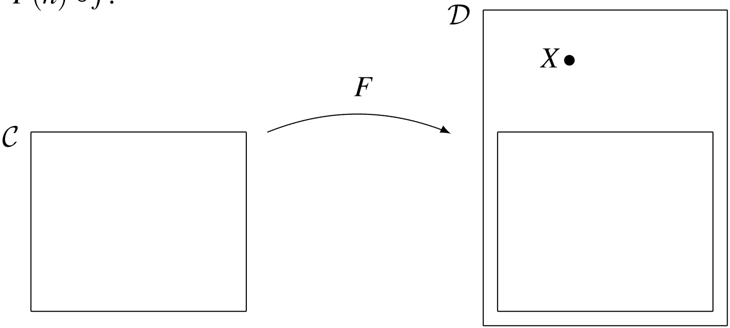
# Universal morphisms

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *universal morphism* **from** $X \in \mathrm{Ob}(\mathcal{D})$ **to** $F$ is a pair $(A, f) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ that is "universal" among all such pairs for the property that $f\colon X \to F(A)$.

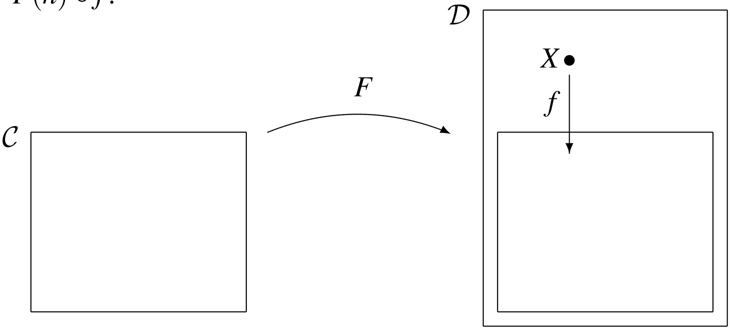
Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *universal morphism* **from** $X \in \mathrm{Ob}(\mathcal{D})$ **to** $F$ is a pair $(A, f) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ that is "universal" among all such pairs for the property that $f\colon X \to F(A)$. The universality means that if $(B, g)$ is another pair with $g\colon X \to F(B)$, then there is a unique $h\colon A \to B$ such that $g = F(h) \circ f$.

# Universal morphisms

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *universal morphism* **from** $X \in \mathrm{Ob}(\mathcal{D})$ **to** $F$ is a pair $(A, f) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ that is "universal" among all such pairs for the property that $f\colon X \to F(A)$. The universality means that if $(B, g)$ is another pair with $g\colon X \to F(B)$, then there is a unique $h\colon A \to B$ such that $g = F(h) \circ f$.

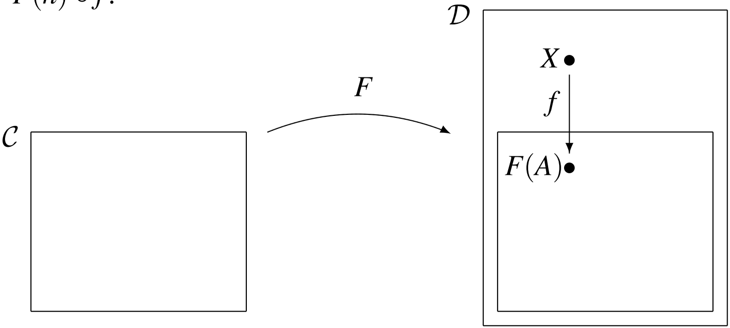


Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *universal morphism* **from** $X \in \mathrm{Ob}(\mathcal{D})$ **to** $F$ is a pair $(A, f) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ that is "universal" among all such pairs for the property that $f\colon X \to F(A)$. The universality means that if $(B, g)$ is another pair with $g\colon X \to F(B)$, then there is a unique $h\colon A \to B$ such that $g = F(h) \circ f$.



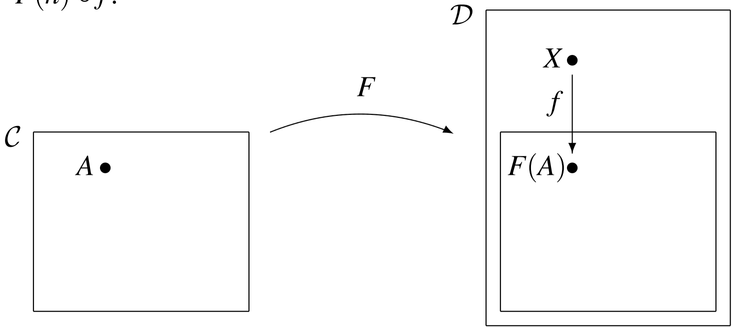
# Universal morphisms

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *universal morphism* **from** $X \in \mathrm{Ob}(\mathcal{D})$ **to** $F$ is a pair $(A,f) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ that is "universal" among all such pairs for the property that $f\colon X \to F(A)$. The universality means that if $(B,g)$ is another pair with $g\colon X \to F(B)$, then there is a unique $h\colon A \to B$ such that $g = F(h) \circ f$.



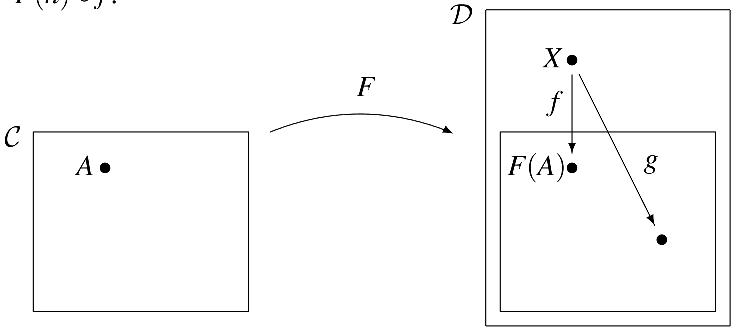
# Universal morphisms

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *universal morphism* **from** $X \in \mathrm{Ob}(\mathcal{D})$ **to** $F$ is a pair $(A, f) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ that is "universal" among all such pairs for the property that $f\colon X \to F(A)$. The universality means that if $(B, g)$ is another pair with $g\colon X \to F(B)$, then there is a unique $h\colon A \to B$ such that $g = F(h) \circ f$.



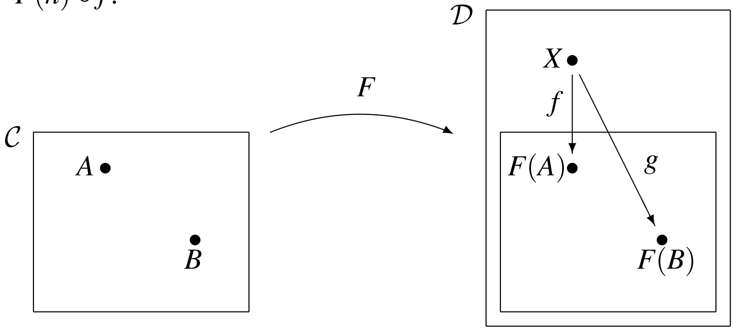
# Universal morphisms

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *universal morphism* **from** $X \in \mathrm{Ob}(\mathcal{D})$ **to** $F$ is a pair $(A, f) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ that is "universal" among all such pairs for the property that $f\colon X \to F(A)$. The universality means that if $(B, g)$ is another pair with $g\colon X \to F(B)$, then there is a unique $h\colon A \to B$ such that $g = F(h) \circ f$.

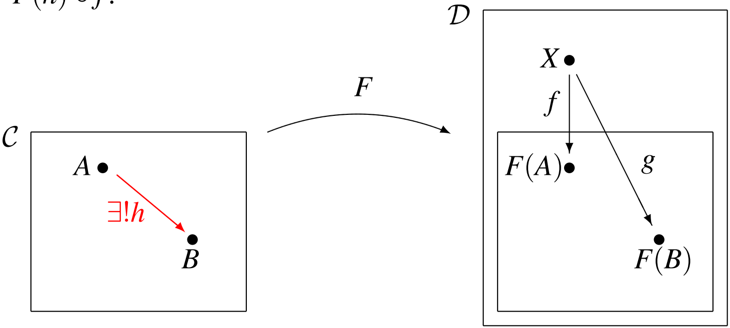


# Universal morphisms

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *universal morphism* **from** $X \in \mathrm{Ob}(\mathcal{D})$ **to** $F$ is a pair $(A, f) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ that is "universal" among all such pairs for the property that $f\colon X \to F(A)$. The universality means that if $(B, g)$ is another pair with $g\colon X \to F(B)$, then there is a unique $h\colon A \to B$ such that $g = F(h) \circ f$.

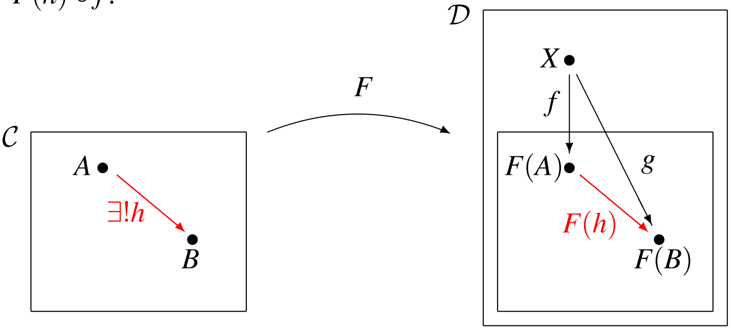


Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *universal morphism* **from** $X \in \mathrm{Ob}(\mathcal{D})$ **to** $F$ is a pair $(A, f) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ that is "universal" among all such pairs for the property that $f\colon X \to F(A)$. The universality means that if $(B, g)$ is another pair with $g\colon X \to F(B)$, then there is a unique $h\colon A \to B$ such that $g = F(h) \circ f$.



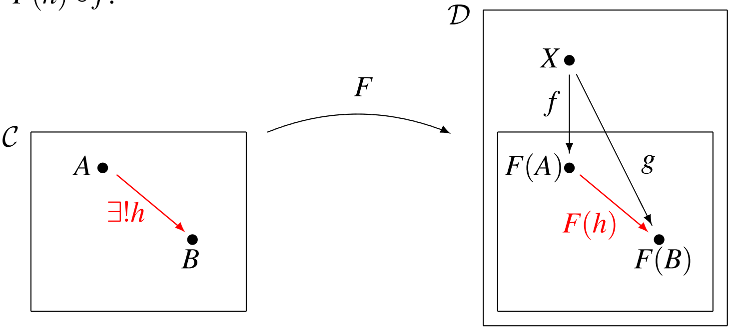
# Universal morphisms

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *universal morphism* **from** $X \in \mathrm{Ob}(\mathcal{D})$ **to** $F$ is a pair $(A, f) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ that is "universal" among all such pairs for the property that $f\colon X \to F(A)$. The universality means that if $(B, g)$ is another pair with $g\colon X \to F(B)$, then there is a unique $h\colon A \to B$ such that $g = F(h) \circ f$.



# Universal morphisms

Given a functor $F\colon \mathcal{C} \to \mathcal{D}$, a *universal morphism* **from** $X \in \mathrm{Ob}(\mathcal{D})$ **to** $F$ is a pair $(A, f) \in \mathrm{Ob}(\mathcal{C}) \times \mathrm{Mor}(\mathcal{D})$ that is "universal" among all such pairs for the property that $f\colon X \to F(A)$. The universality means that if $(B, g)$ is another pair with $g\colon X \to F(B)$, then there is a unique $h\colon A \to B$ such that $g = F(h) \circ f$.




For universal morphism **to** $X$, **from** $F$ reverse the directions of the morphisms.

# Universal properties

# Universal properties

The statement that $(A, f)$ is a universal morphism is the *universal property* of this pair.

# Universal properties

The statement that $(A, f)$ is a universal morphism is the *universal property* of this pair.

**Examples.**

# Universal properties

The statement that $(A, f)$ is a universal morphism is the *universal property* of this pair.

**Examples.**

1. Let $\mathcal{C} \times \mathcal{C}$ be the category whose objects are pairs $(A, B)$ where $A, B \in \mathrm{Ob}(\mathcal{C})$ and whose morphisms are pairs $(f, g)$ where $f, g \in \mathrm{Mor}(\mathcal{C})$.

The statement that $(A, f)$ is a universal morphism is the *universal property* of this pair.

**Examples.**

1. Let $\mathcal{C} \times \mathcal{C}$ be the category whose objects are pairs $(A, B)$ where $A, B \in \mathrm{Ob}(\mathcal{C})$ and whose morphisms are pairs $(f, g)$ where $f, g \in \mathrm{Mor}(\mathcal{C})$.

# Universal properties

The statement that $(A, f)$ is a universal morphism is the *universal property* of this pair.

**Examples.**

1. Let $\mathcal{C} \times \mathcal{C}$ be the category whose objects are pairs $(A, B)$ where $A, B \in \mathrm{Ob}(\mathcal{C})$ and whose morphisms are pairs $(f, g)$ where $f, g \in \mathrm{Mor}(\mathcal{C})$. Now let $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the functor $C \mapsto (C, C)$, $e \mapsto (e, e)$.

The statement that $(A, f)$ is a universal morphism is the *universal property* of this pair.

**Examples.**

1. Let $\mathcal{C} \times \mathcal{C}$ be the category whose objects are pairs $(A, B)$ where $A, B \in \mathrm{Ob}(\mathcal{C})$ and whose morphisms are pairs $(f, g)$ where $f, g \in \mathrm{Mor}(\mathcal{C})$. Now let $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the functor $C \mapsto (C, C)$, $e \mapsto (e, e)$. For any $X = (A, B)$ in this category, a universal morphism **from** $\Delta$ **to** $X$ is a *product* $(P, (\pi_A, \pi_B))$ of $A$ and $B$.

The statement that $(A, f)$ is a universal morphism is the *universal property* of this pair.

**Examples.**

1. Let $\mathcal{C} \times \mathcal{C}$ be the category whose objects are pairs $(A, B)$ where $A, B \in \mathrm{Ob}(\mathcal{C})$ and whose morphisms are pairs $(f, g)$ where $f, g \in \mathrm{Mor}(\mathcal{C})$. Now let $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the functor $C \mapsto (C, C)$, $e \mapsto (e, e)$. For any $X = (A, B)$ in this category, a universal morphism **from** $\Delta$ **to** $X$ is a *product* $(P, (\pi_A, \pi_B))$ of $A$ and $B$.
2. A universal morphism **to** $\Delta$ **from** $X$ is a *coproduct* of $A$ and $B$.

# Universal properties

The statement that $(A, f)$ is a universal morphism is the *universal property* of this pair.

**Examples.**

1. Let $\mathcal{C} \times \mathcal{C}$ be the category whose objects are pairs $(A, B)$ where $A, B \in \mathrm{Ob}(\mathcal{C})$ and whose morphisms are pairs $(f, g)$ where $f, g \in \mathrm{Mor}(\mathcal{C})$. Now let $\Delta \colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the functor $C \mapsto (C, C)$, $e \mapsto (e, e)$. For any $X = (A, B)$ in this category, a universal morphism **from** $\Delta$ **to** $X$ is a *product* $(P, (\pi_A, \pi_B))$ of $A$ and $B$.
2. A universal morphism **to** $\Delta$ **from** $X$ is a *coproduct* of $A$ and $B$.

The statement that $(A, f)$ is a universal morphism is the *universal property* of this pair.

**Examples.**

1. Let $\mathcal{C} \times \mathcal{C}$ be the category whose objects are pairs $(A, B)$ where $A, B \in \mathrm{Ob}(\mathcal{C})$ and whose morphisms are pairs $(f, g)$ where $f, g \in \mathrm{Mor}(\mathcal{C})$. Now let $\Delta \colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ be the functor $C \mapsto (C, C)$, $e \mapsto (e, e)$. For any $X = (A, B)$ in this category, a universal morphism **from** $\Delta$ **to** $X$ is a *product* $(P, (\pi_A, \pi_B))$ of $A$ and $B$.
2. A universal morphism **to** $\Delta$ **from** $X$ is a *coproduct* of $A$ and $B$.