INTRODUCTION TO TENSOR PRODUCTS

The $\otimes$ concept. Let $\mathcal{U}, V$ and $\mathcal{W}$ be varieties of algebras. Let $F := \mathrm{Hom}_{\mathcal{U}}(A, \_\_)$ and $G := \mathrm{Hom}_{\mathcal{V}}(B, \_\_)$ be representable functors[1] between them:

$$\mathcal{U} \xrightarrow{F} \mathcal{V} \xrightarrow{G} \mathcal{W}.$$

Theorem 4 of [2] proves that the composition of representable functors is representable,[2] so there is an object T of $\mathcal{U}$ and a natural isomorphism

$$\mathrm{Hom}_{\mathcal{U}}(T, X) \cong G \circ F(X) = \mathrm{Hom}_{\mathcal{V}}(B, \mathrm{Hom}_{\mathcal{U}}(A, X)). \tag{0.1}$$

The object T is unique up to isomorphism, and it is called the *tensor product* of B and A and is denoted $B \otimes A$.

Summary: Tensor products exist to describe the composition of representable functors.

Background Information.

Representable functors. As mentioned in Example 1 of the handout *Categories and Functors* and in Definition 4 of handout *Equivalence of Categories*, a covariant functor $F \colon \mathcal{U} \to \mathcal{V}$ is said to be *representable* if it is naturally isomorphic to a functor of the form $\mathrm{Hom}_{\mathcal{U}}(A, \_\_)$ for some object A of $\mathcal{U}$. The problem of identifying which functors between varieties are representable was solved by Freyd in [2]. One version of his theorem is:

Theorem 1. *The following are equivalent for a functor F from a variety $\mathcal{U}$ to a variety $\mathcal{V}$:*

1. *F is representable.*
2. *F preserves* products *and* equalizers.[3]
3. *F is* continuous.[4]
4. *F has a* left adjoint.[5]

For this note, the only important part of this theorem is that items (2)–(4) are easily shown to be properties of functors that are preserved under composition, so this theorem justifies our earlier remark that the composition of representable functors is representable.

An illustrative example of a representable functor is the functor $\mathrm{SL}_2 \colon \mathcal{CR} \to \mathcal{SG}$ where $\mathcal{CR}$ is the variety of commutative rings and $\mathcal{SG}$ is the variety of semigroups. This is the

[1] I am restricting these categories to varieties of algebras because representable functors between arbitrary categories need not exist, in general. The values of a representable functor must have underlying sets. Representable functors arise frequently and naturally between varieties, and between some other categories.

[2] The proof that the composition of representable functors is representable makes use of free algebras, the existence of which depends nontrivially on the assumption that $\mathcal{U}, V$ and $\mathcal{W}$ are varieties.

[3] Products and equalizers are examples of categorical constructions called *limits*. All limits can be constructed from products and equalizers, and we have discussed categorical products before. The *equalizer* of a set of $\mathcal{C}$-morphisms $f_i \colon A \to B$ is a $\mathcal{C}$-morphism $e \colon E \to A$ such that $f_i \circ e = f_j \circ e$ for all i and j, and which is universal for this property. Universality means that given a $\mathcal{C}$-morphism $d \colon D \to A$ satisfying $f_i \circ d = f_j \circ d$ for all i and j, there exists a unique $c \colon D \to E$ such that $d = e \circ c$.

[4] I.e., preserves all limits.

[5] A functor $F^* \colon \mathcal{D} \to \mathcal{C}$ is a *left adjoint* of $F \colon \mathcal{C} \to \mathcal{D}$ if there is a natural isomorphism between bifunctors $\mathrm{Hom}_{\mathcal{C}}(F^*(X), Y) \cong \mathrm{Hom}_{\mathcal{D}}(X, F(Y))$. If $\mathcal{C}$ and $\mathcal{D}$ are varieties, and F^* is the left adjoint for F, then an object representing F is $F^*(\mathbf{F}_{\mathcal{D}}(1))$, then F^*-image of the 1-generated free algebra of $\mathcal{D}$.

functor that assigns to a commutative ring S the multiplicative semigroup of 2×2 matrices of determinant 1 with entries in S. Of course, $\mathrm{SL}_2(S)$ is even a group, but for simplicity I want to discuss it as an object defined with only one operation.

The functor $\mathrm{SL}_2(\_\_)$ is represented by the commutative ring with the presentation

$$\langle G \mid R \rangle = \left\langle w, x, y, z \;\middle|\; \det \begin{bmatrix} w & x \\ y & z \end{bmatrix} = 1 \right\rangle, \tag{0.2}$$

which is the ring $A = \mathbb{Z}[G]/(R) = \mathbb{Z}[w,x,y,z]/(wz - yx - 1)$. One may view this as the commutative ring over which there is a generic 2×2-matrix of determinant 1. The universal property of presentations says that a homomorphism from A to a commutative ring S is specified by a 4-tuple of elements of S satisfying the same relations as those satisfied by the generators in the presentation (0.2); namely, a tuple $(a, b, c, d) \in S^4$ satisfying $ad - cb = 1$. Thus, the elements of $\mathrm{SL}_2(S)$ are in 1-1 correspondence with the elements of $\mathrm{Hom}_{\mathcal{CR}}(A, S)$, indicating that $\mathrm{Hom}_{\mathcal{CR}}(A, \_\_)$ represents (the underlying set part of) the functor SL_2.

Of course, if S is a commutative ring, then $\mathrm{SL}_2(S)$ is more than a set, it is a semigroup. It is necessary to explain next how to equip each hom-set $\mathrm{Hom}_{\mathcal{CR}}(A, S)$ with semigroup structure "in a uniform way". The amount of uniformity required is "enough to make $\mathrm{Hom}_{\mathcal{CR}}(A, \_\_)$ a semigroup-valued functor." Here is what is required to make this true. Choose a semigroup multiplication on $\mathrm{Hom}_{\mathcal{CR}}(A, S)$ for each commutative ring S. This is one associative function

$$m_S \colon \mathrm{Hom}_{\mathcal{CR}}(A, S)^2 \to \mathrm{Hom}_{\mathcal{CR}}(A, S)$$

for each S. The choices should be made so that each $\mathcal{CR}$-morphism $h \colon S \to T$ induces a homomorphism from $\langle \mathrm{Hom}_{\mathcal{CR}}(A, S); m_S \rangle$ to $\langle \mathrm{Hom}_{\mathcal{CR}}(A, T); m_T \rangle$, which means that for all h, S and T there is a commutative square of set functions:

$$\begin{array}{ccc}
\mathrm{Hom}_{\mathcal{CR}}(A, S)^2 & \xrightarrow{m_S} & \mathrm{Hom}_{\mathcal{CR}}(A, S) \\
{\scriptstyle h^2}\big\downarrow & & {\scriptstyle h}\big\downarrow \\
\mathrm{Hom}_{\mathcal{CR}}(A, T)^2 & \xrightarrow{m_T} & \mathrm{Hom}_{\mathcal{CR}}(A, T)
\end{array}$$

In other words, the functions $(m_S)_{S \in \mathrm{Ob}(\mathcal{CR})}$ constitute a natural transformation from the functor $\mathrm{Hom}_{\mathcal{CR}}(A, \_\_)^2$ to the functor $\mathrm{Hom}_{\mathcal{CR}}(A, \_\_)$. It is suggestive to replace the first functor $\mathrm{Hom}_{\mathcal{CR}}(A, \_\_)^2$ with the naturally isomorphic[6] functor $\mathrm{Hom}_{\mathcal{CR}}(A \sqcup A, \_\_)$, which is represented by $A \sqcup A$. Thus, the problem of equipping the hom-sets $\mathrm{Hom}_{\mathcal{CR}}(A, \_\_)$ with an associative multiplication in such a way as to make this a functor to a variety of semigroups is identical to the problem of finding an "associative" natural transformation between the representable set-valued functors $m \colon \mathrm{Hom}_{\mathcal{CR}}(A \sqcup A, \_\_) \to \mathrm{Hom}_{\mathcal{CR}}(A, \_\_)$. This problem is solved by the Yoneda Lemma.

The Yoneda Lemma. The Yoneda Lemma (Lemma III.2.1 of [3]) classifies the natural transformations from a representable functor to any other set-valued functor. The following corollary is the statement in the case that both functors are representable.

[6]The fact that $\mathrm{Hom}_{\mathcal{CR}}(A, \_\_)^2$ is naturally isomorphic to $\mathrm{Hom}_{\mathcal{CR}}(A \sqcup A, \_\_)$ follows from the universal property of coproducts.

Corollary 2. (Yoneda) *Suppose that* $F = \mathrm{Hom}_{\mathcal{C}}(A, \_\_)$ *and* $G = \mathrm{Hom}_{\mathcal{C}}(B, \_\_)$ *are representable functors* $F, G\colon \mathcal{C} \to \mathcal{SET}$. *If* $f\colon B \to A$ *is a* $\mathcal{C}$*-morphism, then*

$$f^*\colon F \to G\colon h \mapsto h \circ f$$

is a natural transformation. Moreover, every *natural transformation from* F *to* G *is of this form.*

Thus, a natural transformation m from $\mathrm{Hom}_{\mathcal{CR}}(A \sqcup A, \_\_)$ to $\mathrm{Hom}_{\mathcal{CR}}(A, \_\_)$ has the form $m = \mu^*$ for some $\mathcal{CR}$-morphism $\mu\colon A \to A \sqcup A$, called *comultiplication*. The specification of such a morphism is exactly what is needed to equip all hom-sets $\mathrm{Hom}_{\mathcal{CR}}(A, \_\_)$ with a semigroup operation to make this into a functor to the variety of semigroups.

To continue our SL_2 example, recall that $A = \mathbb{Z}[w, x, y, z]/(wz - yx - 1)$. Since the coproduct of the objects given by presentations $\langle G_1 \mid R_1 \rangle$ and $\langle G_2 \mid R_2 \rangle$ is the object presented by the union $\langle G_1 \cup G_2 \mid R_1 \cup R_2 \rangle$, it follows that

$$\begin{aligned} A \sqcup A &= \langle w, x, y, z, w', x', y', z' \mid wz - yx = 1, w'z' - y'x' = 1 \rangle \\ &= \mathbb{Z}[w, x, y, z, w', x', y', z']/(wz - yx - 1, w'z' - y'x' - 1). \end{aligned}$$

In other words, $A \sqcup A$ is the commutative ring over which there are two generic 2×2-matrices of determinant 1. Over this ring, there is a generic product of two 2×2-matrices of determinant 1, it is

$$\begin{bmatrix} w & x \\ y & z \end{bmatrix} \cdot \begin{bmatrix} w' & x' \\ y' & z' \end{bmatrix} = \begin{bmatrix} ww' + xy' & wx' + xz' \\ yw' + zy' & yx' + zz' \end{bmatrix}.$$

The homomorphism

$$\mu\colon A \to A \sqcup A\colon (w, x, y, z) \mapsto (ww' + xy', wx' + xz', yw' + zy', yx' + zz'),$$

which selects the generic product of 2×2-matrices, is the comultiplication that induces the usual matrix multiplication on all hom-sets. The general discussion of this process requires the notion of a *coalgebra*.

Coalgebras. For simplicity, I will concentrate on cogroups and assume that it is then obvious how to define corings, comodules, etc. (or see [1]).

Definition 3. Let $\mathcal{C}$ be a category with finite products. A *group in* $\mathcal{C}$ is an object $G \in \mathrm{Ob}(\mathcal{C})$ equipped with morphisms $m\colon G^2 \to G$ (multiplication), $i\colon G \to G$ (inverse), and $e\colon G \to G$ (unit)[7] satisfying "the group identities". These are: the associative law, the unit law, the law for inverses, and the law stating that the e-morphism is a constant. These laws are expressible with commutative diagrams, as follows:

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times \mathrm{id}} & G \times G \\ {\scriptstyle \mathrm{id} \times m}\downarrow & & \downarrow{\scriptstyle m} \\ G \times G & \xrightarrow{m} & G \end{array} \qquad\qquad \begin{array}{ccc} G \times G & \xrightarrow{\pi_1} & G \\ {\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle e} \\ G & \xrightarrow{e} & G \end{array}$$

[7] For coalgebras, it is my convention that distinguished constants will be constant unary functions rather than zeroary functions. This prevents the exclusion of interesting examples for trivial reasons.

(The associativity of m is expressed on the left, and the constancy of e on the right.)

$$\begin{tikzcd}
G \arrow[r, "e\times\mathrm{id}"] \arrow[d, "\mathrm{id}"'] & G\times G \arrow[d, "m"'] & G \arrow[l, "\mathrm{id}\times e"'] \arrow[d, "\mathrm{id}"'] \\
G \arrow[r, equal] & G \arrow[r, equal] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \arrow[r, "i\times\mathrm{id}"] \arrow[d, "e"'] & G\times G \arrow[d, "m"'] & G \arrow[l, "\mathrm{id}\times i"'] \arrow[d, "e"'] \\
G \arrow[r, equal] & G \arrow[r, equal] & G
\end{tikzcd}$$

(The unit laws are expressed on the left, and the inverse laws on the right.)

For example, a group in $\mathcal{SET}$ is just an ordinary group. Next, dualize:

Definition 4. Let $\mathcal{C}$ be a category with finite coproducts. A *cogroup in* $\mathcal{C}$ is an object $G \in \mathrm{Ob}(\mathcal{C})$ equipped with morphisms $m\colon G \to G \sqcup G$ (comultiplication), $i\colon G \to G$ (coinverse), and $e\colon G \to G$ (coidentity), satisfying the *coidentities* that are the duals of the groups identities:

$$\begin{tikzcd}
G\sqcup G\sqcup G & G\sqcup G \arrow[l, "m\sqcup\mathrm{id}"'] \\
G\sqcup G \arrow[u, "\mathrm{id}\sqcup m"] & G \arrow[l, "m"'] \arrow[u, "m"']
\end{tikzcd}
\qquad
\begin{tikzcd}
G\sqcup G & G \arrow[l, "\iota_1"'] \\
G \arrow[u, "\iota_2"] & G \arrow[l, "e"'] \arrow[u, "e"']
\end{tikzcd}$$

(coassociativity and coconstancy), and

$$\begin{tikzcd}
G & G\sqcup G \arrow[l, "e\sqcup\mathrm{id}"'] \arrow[r, "\mathrm{id}\sqcup e"] & G \\
G \arrow[u, "\mathrm{id}"] \arrow[r, equal] & G \arrow[u, "m"] \arrow[r, equal] & G \arrow[u, "\mathrm{id}"]
\end{tikzcd}
\qquad
\begin{tikzcd}
G & G\sqcup G \arrow[l, "i\sqcup\mathrm{id}"'] \arrow[r, "\mathrm{id}\sqcup i"] & G \\
G \arrow[u, "e"] \arrow[r, equal] & G \arrow[u, "m"] \arrow[r, equal] & G \arrow[u, "e"]
\end{tikzcd}$$

(counit unit laws and coinverse laws).

The theorem that is important for us, and which follows from the Yoneda Lemma (as discussed above), is:

Theorem 5. *Let $\mathcal{U}$ and $\mathcal{V}$ be varieties. The following are equivalent.*

(1) *A represents a covariant functor from $\mathcal{U}$ to $\mathcal{V}$.*
(2) *A is a $\mathcal{U}$-algebra that is simultaneously a $\mathcal{V}$-coalgebra.*

I will write $A_{\mathcal{U}}$ to indicate that A is an algebra in $\mathcal{U}$, and ${}_{\mathcal{V}}A$ to indicate that A is a $\mathcal{V}$-coalgebra. Thus, ${}_{\mathcal{V}}A_{\mathcal{U}}$ indicates that A is a $\mathcal{U}$-algebra that is simultaneously a $\mathcal{V}$-coalgebra.

Example 6. (Spheres are cogroups.) The category in question is the category **TopH** whose objects are topological spaces with base point and whose morphisms are the homotopy classes of continuous functions. If $S = S^n$ is the (pointed) n-sphere, $n > 0$, then the coproduct $S \sqcup S$ is the space obtained by taking two disjoint copies of S and identifying base points. There is a comultiplication $m\colon S \to S \sqcup S$ defined as follows: consider a great circle on S that passes through the base point. Gradually contract the great circle to a point, collapsing each hemisphere of S to a sphere with half the surface area, thereby obtaining a space homeomorphic to $S \sqcup S$. This map is m. The coinverse can be taken to be any orientation-reversing homeomorphism, and the coidentity is the constant function whose range is the base point.

The cooperations do not satisfy the cogroup coidentities in **Top**, the category of topological spaces and continuous maps, but satisfy them in **TopH**. I.e., although $(m \times \mathrm{id}) \circ m$ and $(\mathrm{id} \times m) \circ m$ are not equal maps, they are homotopic. This indicates the reason for working in **TopH**.

The functor represented by S^n is $\pi_n(\_\_) := \mathrm{Hom}_{\mathbf{TopH}}(S^n, \_\_)$, the n-th homotopy group functor.

Tensor product of objects once again. I now return to the explanation, from the first paragraph of this note, of the tensor product concept.

Theorem 4 of [2] proves that the composition of representable functors $F := \mathrm{Hom}_{\mathcal{U}}(A, \_\_)$ and $G := \mathrm{Hom}_{\mathcal{V}}(B, \_\_)$ is representable. Equivalently, if $A = {}_{\mathcal{V}}A_{\mathcal{U}}$ and $B = {}_{\mathcal{W}}B_{\mathcal{V}}$, then

(i) a representor for $G \circ F$ exists, and it is $B \otimes A$. Moreover,
(ii) $B \otimes A = ({}_{\mathcal{W}}B_{\mathcal{V}}) \otimes ({}_{\mathcal{V}}A_{\mathcal{U}}) = {}_{\mathcal{W}}(B_{\mathcal{V}} \otimes {}_{\mathcal{V}}A)_{\mathcal{U}} = {}_{\mathcal{W}}(B \otimes A)_{\mathcal{U}}$ is a $\mathcal{U}$-algebra that is simultaneously a $\mathcal{W}$-coalgebra.

This tells us exactly when the tensor product of two algebras (typically from different varieties) exists. The next question, *"How do you construct $B \otimes A$?"*, is answered by Theorem 6 of [2]. It is possible to write down a presentation for it. The presentation has the form $\langle G \mid R \rangle$ where the set of generators is $G = B \times A$, written in the "simple tensor notation", $B \times A = \{b \otimes a \mid b \in B, a \in A\}$. The relations are of two types:

Theorem 7. (A presentation of $B \otimes A$.) *Let $A = {}_{\mathcal{V}}A_{\mathcal{U}}$ and $B = B_{\mathcal{V}}$. Then $B \otimes A$ is the $\mathcal{U}$-algebra with the presentation $\langle G \mid R_1 \cup R_2 \rangle$ where $G = B \times A$ and relations*

(R_1) *If f is an m-ary $\mathcal{U}$-operation, $b \in B$ and $a_1, \ldots, a_m \in A$, then*

$$b \otimes f(a_1, \ldots, a_m) = f(b \otimes a_1, \ldots, b \otimes a_m).$$

(R_2) *If g is an n-ary $\mathcal{V}$-operation, $b_1, \ldots, b_n \in B$, $a \in A$, and the corresponding the n-ary co-operation $g \colon A \to \sqcup^n A$ satisfies $g(a) = h(a_1, \ldots, a_n)$, then*

$$g(b_1, \ldots, b_n) \otimes a = h(b_1 \otimes a_1, \ldots, b_n \otimes a_n).$$

This presentation can be used to deduce some basic properties of the tensor product of algebras, namely

(i) (Tensor product distributes over coproduct in each variable.) $A \otimes (\coprod_{i \in I} B_i) \cong \coprod_{i \in I}(A \otimes B_i)$ and $(\coprod_{i \in I} A_i) \otimes B \cong \coprod_{i \in I}(A_i \otimes B)$.
(ii) Let $\mathbf{F}_{\mathcal{V}}(X)$ denote the free $\mathcal{V}$-algebra over the set X. If $B \in \mathcal{V}$, then $B \otimes \mathbf{F}_{\mathcal{V}}(1) \cong B$. If $A \in \mathcal{V}$ and A is also a $\mathcal{V}$-coalgebra, then $\mathbf{F}_{\mathcal{V}}(1) \otimes A \cong A$.
(iii) (Follows from (i) and (ii), and the fact that free algebras in $\mathcal{V}$ are $\mathcal{V}$-coalgebras.) $\mathbf{F}_{\mathcal{V}}(X) \otimes \mathbf{F}_{\mathcal{V}}(Y) \cong \mathbf{F}_{\mathcal{V}}(X \times Y)$.

Example 8. (Tensor products of sets.) If X is an object in $\mathcal{SET}$, then $X = \mathbf{F}_{\mathcal{SET}}(X)$. Hence, by (iii) above, $X \otimes Y = X \times Y$. Therefore, tensor product and binary Cartesian product coincide in the category of sets.

Example 9. (Tensor products of abelian groups.) Every abelian group is also a abelian cogroup. Indeed, any module is an abelian cogroup. Recall that a comultiplication is a homomorphism $m\colon A \to A \sqcup A$. But in a variety with a biproduct one has $A \sqcup A = A \oplus A = A \sqcap A$. Thus, the diagonal homomorphism $\Delta\colon A \to A \sqcap A = A \sqcup A$ is a comultiplication. The properties of the biproduct guarantee that this is an abelian cogroup comultiplication.

The structure of the tensor product of two abelian groups may be determined by the presentation of Theorem 7. But, it is already known to you: $B \otimes A = \langle B \times A \mid R \rangle$ where R is the set of all relations expressing the biadditivity of $\otimes$. Namely, for all $a, a' \in A, b, b' \in B$ one has $b \otimes (a + a') = b \otimes a + b \otimes a'$ (a relation of type R_1 from Theorem 7) and $(b + b') \otimes a = b \otimes a + b' \otimes a$ (a relation of type R_2).

Example 10. (Tensor products of $\mathbf{R}$-modules/$\mathbb{F}$-spaces.) A representable functor from the variety of right $\mathbf{R}$-modules to the category of right $\mathbf{S}$-modules is represented by a module ${}_{\mathbf{S}}A_{\mathbf{R}}$ that is simultaneously a right $\mathbf{R}$-module and a right $\mathbf{S}$-comodule. Such a structure is nothing other than an $\mathbf{S}, \mathbf{R}$-bimodule. That is, a right $\mathbf{S}$-comodule structure on the right $\mathbf{R}$-module A is a left $\mathbf{S}$-module structure on A that is compatible with the right $\mathbf{R}$-module structure in the sense of the bimodule identity $\forall s \in S, \forall a \in A, \forall r \in R((sa)r = s(ar))$.

Thus, if $A = {}_{\mathbf{S}}A_{\mathbf{R}}$ is an $\mathbf{S}, \mathbf{R}$-bimodule and $B = {}_{\mathbf{T}}B_{\mathbf{S}}$ is an $\mathbf{T}, \mathbf{S}$-bimodule, then $B \otimes A = {}_{\mathbf{T}}(B \otimes A)_{\mathbf{R}}$ is an $\mathbf{T}, \mathbf{R}$-bimodule. The presentation for $B \otimes A$ is $\langle B \times A \mid R \rangle$ where R expresses biadditivity and middle linearity with respect to $\mathbf{S}$: for all $a, a' \in A, b, b' \in B, s \in S$ one has $b \otimes (a + a') = b \otimes a + b \otimes a'$, $(b + b') \otimes a = b \otimes a + b' \otimes a$ and $(as) \otimes b = a \otimes (sb)$.

If $\mathbf{R}$ is a commutative ring, then any right $\mathbf{R}$-module is also an $\mathbf{R}, \mathbf{R}$-bimodule, hence is simultaneously a right $\mathbf{R}$-module and a left $\mathbf{R}$-comodule. Thus, the tensor product of any two right $\mathbf{R}$-modules exists and is a right $\mathbf{R}$-module. It is given by the presentation of the previous paragraph as the case where $\mathbf{R} = \mathbf{S} = \mathbf{T}$.

If $\mathbb{F}$ is a field, then any $\mathbb{F}$-module is free. Hence, if U and V are $\mathbb{F}$-spaces with bases X and Y, then $U \otimes V$ is a space with basis $X \times Y$.

Example 11. (Tensor products of entropic algebras.) If f is m-ary and g is n-ary, then the the identity

$$f(g(x_{11}, \ldots, x_{1n}), \ldots, g(x_{m1}, \ldots, x_{mn})) = g(f(x_{11}, \ldots, x_{m1}), \ldots, f(x_{1n}, \ldots, x_{mn}))$$

expresses that f and g *commute* with each other. If the identities defining a variety $\mathcal{V}$ imply that all of the operations commute with each other, then $\mathcal{V}$ is *entropic*. The varieties of sets, abelian groups, and $\mathbb{F}$-spaces are entropic. If $\mathcal{V}$ is entropic, then every $\mathcal{V}$-algebra is naturally a $\mathcal{V}$-coalgebra. Hence, the tensor product of any two algebras in $\mathcal{V}$ exists, and belongs to $\mathcal{V}$. The structure of $B \otimes A$ is given by the presentation in Theorem 7.

Tensor product of morphisms. I have been describing the tensor product of objects; I now turn to the tensor product of morphisms.

Suppose that $f\colon A \to A'$ in $\mathcal{U}$, $g\colon B \to B'$ in $\mathcal{V}$, and that both $B \otimes A$ and $B' \otimes A'$ exist (which will happen if A and A' are $\mathcal{V}$-coalgebras). Then f and g induce a natural morphism $g \otimes f\colon B \otimes A \to B' \otimes A'$, whose existence I now justify. An element $\varphi \in \mathrm{Hom}_{\mathcal{V}}(B, \mathrm{Hom}_{\mathcal{U}}(A, X))$ is a $\mathcal{V}$-morphism with domain B such that if $b \in B$ then $\varphi(b)$ is a $\mathcal{U}$-morphism from A to X; thus, φ may be considered to be a function $\varphi(y)(x)$ of two variables (with $y \in B \in \mathcal{V}$ and $x \in A \in \mathcal{U}$). By the Yoneda Lemma, the morphism $g\colon B \to B'$

determines natural transformations

$$g^*\colon \mathrm{Hom}_{\mathcal{V}}(B', \mathrm{Hom}_{\mathcal{U}}(A, \_\_)) \to \mathrm{Hom}_{\mathcal{V}}(B, \mathrm{Hom}_{\mathcal{U}}(A, \_\_))\colon \varphi(y)(x) \mapsto \varphi(g(y))(x)$$

and

$$g^*\colon \mathrm{Hom}_{\mathcal{V}}(B', \mathrm{Hom}_{\mathcal{U}}(A', \_\_)) \to \mathrm{Hom}_{\mathcal{V}}(B, \mathrm{Hom}_{\mathcal{U}}(A', \_\_))\colon \varphi(y)(x) \mapsto \varphi(g(y))(x).$$

Similarly, $f\colon A \to A'$ determines natural transformations

$$f^*\colon \mathrm{Hom}_{\mathcal{V}}(B, \mathrm{Hom}_{\mathcal{U}}(A', \_\_)) \to \mathrm{Hom}_{\mathcal{V}}(B, \mathrm{Hom}_{\mathcal{U}}(A, \_\_))\colon \varphi(y)(x) \mapsto \varphi(y)(f(x))$$

and

$$f^*\colon \mathrm{Hom}_{\mathcal{V}}(B', \mathrm{Hom}_{\mathcal{U}}(A', \_\_)) \to \mathrm{Hom}_{\mathcal{V}}(B', \mathrm{Hom}_{\mathcal{U}}(A, \_\_))\colon \varphi(y)(x) \mapsto \varphi(y)(f(x)).$$

Observe that f^* commutes with g^*, so their composition $f^* \circ g^* = g^* \circ f^*$ is uniquely determined: it is the natural transformation that assigns to $\varphi(y)(x)$ the morphism $\varphi(g(y), f(x))$. This yields a natural transformation from $\mathrm{Hom}_{\mathcal{V}}(B', \mathrm{Hom}_{\mathcal{U}}(A', \_))$ to $\mathrm{Hom}_{\mathcal{V}}(B, \mathrm{Hom}_{\mathcal{U}}(A, \_))$, hence (via (0.1)) a natural transformation from $\mathrm{Hom}_{\mathcal{U}}(B' \otimes A', \_\_)$ to $\mathrm{Hom}_{\mathcal{U}}(B \otimes A, \_\_))$. By the Yoneda Lemma again, any such natural transformation is induced by a unique $\mathcal{U}$-morphism from $B \otimes A$ to $B' \otimes A'$. This is the morphism called $g \otimes f$.[8]

I emphasize that $g \otimes f$ is simply a notation for this natural morphism, and $g \otimes f$ is not assumed to be an element of the tensor product of some pair of objects.

Example 12. (Tensor products of monoids.) Monoids are algebraic models of structures of the form $\mathrm{End}_{\mathcal{SET}}(X)$ — sets of endomorphisms of objects in $\mathcal{SET}$. Within $\mathrm{End}_{\mathcal{SET}}(X \otimes Y) = \mathrm{End}_{\mathcal{SET}}(X \times Y)$ are the elements $f \otimes g$ described above ($f \in \mathrm{End}(X)$, $g \in \mathrm{End}(Y)$). These elements generate a submonoid of $\mathrm{End}(X \otimes Y)$, which may be called the *tensor product* of $\mathrm{End}(X)$ and $\mathrm{End}(Y)$ even though it is not the "true" tensor product of these monoids in the variety of monoids.

More generally, this notion of the tensor product is defined for any two monoids. Given M and N, $M \otimes N$ is the monoid consisting of all pairs $M \times N = \{m \otimes n \mid m \in M, n \in N\}$ with coordinatewise multiplication $(m \otimes n)(m' \otimes n') = mm' \otimes nn'$ and identity $1 \otimes 1$. Up to isomorphism, this is just the ordinary product of M and N, but it has a different universal property. Namely, $M \otimes N$ is equipped with coprojections (not projections!) $i_M\colon M \to M \otimes N\colon m \mapsto m \otimes 1$ and $i_N\colon N \to M \otimes N\colon n \mapsto 1 \otimes n$ such that if $\alpha\colon M \to P$ and $\beta\colon N \to P$ are monoid homomorphisms whose images commute, then there is a unique monoid homomorphism $\alpha \otimes \beta\colon M \otimes N \to P$ such that $\alpha = (\alpha \otimes \beta) \circ i_M$ and $\beta = (\alpha \otimes \beta) \circ i_N$.

Example 13. (Tensor products of rings.) Rings are algebraic modules of endomorphisms of abelian groups. What is typically called the "tensor product of rings" is not the tensor product of two objects in the variety of rings, but is defined in terms of the tensor product of morphisms in the variety of abelian groups. The notion is intended to describe the subring of $\mathrm{End}_{\mathbb{Z}}(A \otimes B)$ that is generated by all $f \otimes g$ ($f \in \mathrm{End}_{\mathbb{Z}}(A)$, $g \in \mathrm{End}_{\mathbb{Z}}(B)$).

In general, given rings $\mathbf{R}$ and $\mathbf{S}$, the ring $\mathbf{R} \otimes \mathbf{S} = \mathbf{R} \otimes_{\mathbb{Z}} \mathbf{S}$ is the ring whose underlying additive group is the tensor product $\langle R; +, -, 0\rangle \otimes \langle S; +, -, 0\rangle$ of the underlying additive

[8] Following the argument through, it is easy to see that morphisms of this type compose coordinatewise, as in $(g \otimes f) \circ (g' \otimes f') = (g \circ g') \otimes (f \circ f')$.

groups of $\mathbf{R}$ and $\mathbf{S}$, and whose multiplication is defined on simple tensors by $(r \otimes s)(r' \otimes s') = rr' \otimes ss'$ and is extended to other elements via the distributive laws.

$\mathbf{R} \otimes \mathbf{S}$ is equipped with coprojections $i_R \colon \mathbf{R} \to \mathbf{R} \otimes \mathbf{S} \colon r \mapsto r \otimes 1$ and $i_S \colon \mathbf{S} \to \mathbf{R} \otimes \mathbf{S} \colon s \mapsto 1 \otimes s$. These are ring homomorphisms. The universal property of the tensor product of rings is that if $\alpha \colon \mathbf{R} \to \mathbf{T}$ and $\beta \colon \mathbf{S} \to \mathbf{T}$ are ring homomorphisms whose images commute, then there is a unique ring homomorphism $\alpha \otimes \beta \colon \mathbf{R} \otimes \mathbf{S} \to \mathbf{T}$ such that $\alpha = (\alpha \otimes \beta) \circ i_R$ and $\beta = (\alpha \otimes \beta) \circ i_T$. If you examine this universal property closely, you will see that, within the variety of commutative rings, $\mathbf{R} \otimes \mathbf{S} = \mathbf{R} \sqcup \mathbf{S}$. (They are defined in terms of the same universal property.)

Example 14. (Tensor products of $\mathbb{F}$-algebras.) $\mathbf{F}$-algebras are algebraic models for $\mathrm{End}_{\mathbb{F}}(V)$ where V is an $\mathbb{F}$-space. Everything said in the previous example has an analogy here. If $\mathbf{A}$ and $\mathbf{B}$ are $\mathbb{F}$-algebras, then $\mathbf{B} \otimes_{\mathbb{F}} \mathbf{A}$ is the $\mathbb{F}$-algebra whose underlying $\mathbb{F}$-space is the tensor product $B \otimes_{\mathbb{F}} A$ of the underlying $\mathbb{F}$-spaces of $\mathbf{B}$ and $\mathbf{A}$. Multiplication is defined coordinatewise on simple tensors, and you can guess the universal property.

The Kronecker product. Suppose that U and V are $\mathbb{F}$-spaces with ordered bases $X = (d_1, \ldots, d_m)$ and $Y = (e_1, \ldots, e_n)$ respectively. Then the lexicographically ordered product $X \times Y = (d_1 \otimes e_1, d_1 \otimes e_2, \ldots, d_m \otimes e_n)$ is an ordered basis for $U \otimes V$. If $f \in \mathrm{End}_{\mathbb{F}}(U)$ and $g \in \mathrm{End}_{\mathbb{F}}(V)$ have matrices $[f] = F = [a_{ij}]$ and $[g] = G = [b_{ij}]$ relative to the ordered bases X and Y, then you can check that the matrix for $f \otimes g$ relative to $X \times Y$ is

$$[f \otimes g] = F \otimes G = \begin{bmatrix} a_{11}G & \cdots & a_{1m}G \\ \vdots & \ddots & \vdots \\ a_{m1}G & \cdots & a_{mn}G \end{bmatrix}.$$

The operation $(F, G) \mapsto F \otimes G$ is called the *Kronecker product of F and G*. It is straightforward to calculate that the eigenvalues of $F \otimes G$ are $\lambda_i \mu_j$ where λ_i is an eigenvalue of F and μ_j is an eigenvalue of G. Thus, $\mathtt{tr}(F \otimes G) = \mathtt{tr}(F) \cdot \mathtt{tr}(G)$.

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