Tensor Product

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Questions. When is the composition of representable functors representable? When so, how do you determine the representing object? (Solve equation for T: $\mathrm{Hom}(T, X) \cong \mathrm{Hom}(B, \mathrm{Hom}(A, X))$.)

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So $\mathrm{Hom}(B, \mathrm{Hom}(A, X)) \cong \mathrm{Hom}(A \times B, X)$.

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So we say $A \otimes B \cong A \times B$ in **Set**.

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is "natural", hence expresses that the functor $A \otimes \_$ is the left adjoint of the hom functor $\operatorname{Hom}(A, \_)$. Since left adjoints are known to preserve colimits it follows that $Z \mapsto A \otimes Z$ will preserve colimits. In particular, $A \otimes (M \oplus N) \cong (A \otimes M) \oplus (A \otimes N)$ for modules.