Spectral spaces

Spec(R)

We take the points of Spec(R) to be (or to be represented by) the prime ideals of R.

We take the closed sets to be the "vanishing sets" of subsets of R:

For $S \subseteq R$, $\quad V(S) = \{\mathfrak{p} \mid S \subseteq \mathfrak{p}\}$ $\quad$ (Why "vanishing"?)

$V(S) = V(\langle S \rangle) = V(r(\langle S \rangle))$. A set is open if it is $V(S)^c$ for some S.

1. $\mathrm{Spec}(R) = V(\{0\})$.
2. $\emptyset = V(\{1\})$.
3. $\bigcap V(S_i) = V(\bigcup S_i)$.
4. $V(I) \cup V(J) = V(IJ) = V(I \cap J)$.
5. In $\mathbb{Z}$, $\bigcup_k V(\{2 \cdot 3 \cdots p_k\})$ is not a vanishing set.

Examples

1. $\mathrm{Spec}(\mathbb{Z})$.
2. $\mathrm{Spec}(k[x])$, k a field.
3. $\mathrm{Spec}(\mathbb{Z}_6[x])$.
4. (Boolean ring, $\mathbb{B} \leq (\mathbb{F}_2)^{\kappa}$) $\mathrm{Spec}(\mathbb{B})$ = Stone space of $\mathbb{B}$.
5. (Nagata idealization, $R \oplus M$) $\mathrm{Spec}(R \oplus M) = \mathrm{Spec}(R)$.

Spectral spaces: Definition & Theorem

Definition. A topological space is *spectral* if it is homeomorphic to $\mathrm{Spec}(R)$ for some commutative ring R. (Don't look at Wikipedia!)

Hochster's Characterization of Spectral Spaces.
A space is spectral iff

1. it is sober,
2. it is compact,
3. it has a basis of compact open sets, and
4. the intersection of two compact open sets is compact open.

Sobriety

Definition. A topological space is *sober* if each closed irreducible subset has a unique generic point.

More primitive concepts:

1. A *topological space* is ....
2. A closed set is *irreducible* if it is (a) not empty and (b) not expressible as the union of two proper closed subsets. That is, C is *irreducible* as a closed set iff it is a *(proper) meet-irreducible* in the lattice of closed sets ordered by reverse inclusion. (If not irreducible, then *reducible*.)
3. A point p is a *generic point* of a closed set C if $\overline{\{p\}} = C$.

Facts/examples/nonexamples.

1. In any topological space, the closure of a point is irreducible. In a sober space, a closed subset is irreducible iff it is the closure of a unique point.
2. Hausdorff spaces are sober. In such spaces, the closed irreducible sets are the singletons.
3. MaxSpec(R) need not be sober. (Max(R), AM; m-Spec(R), Mat.)

Compact open $\leftrightarrow V(r_1, \ldots, r_k)^c \leftrightarrow \bigcup_{\text{fin}}$ distinguished

Let's discuss:

1. Compact subsets of a space.
2. Compact elements of a lattice.
3. Compact elements of the lattice of closed sets of an algebraic closure operator.
4. Compact elements of $\mathrm{Idl}(R)$; of the associated frame.
5. **Thm.** An element is compact in the lattice of radical ideals iff it is the radical of a finitely generated ideal $r(\langle f_1, \ldots, f_k \rangle)$.
6. A compact open subset of $\mathrm{Spec}(R)$ is a finite union of "distinguished" (or "principal") open sets:

   $$D(f) := V(f)^c = \{\mathfrak{p} \mid f \notin \mathfrak{p}\}. \quad \text{(Think: } D(f) = \text{support of } f.\text{)}$$

   equivalently it has the form $V(f_1, \ldots, f_k)^c$, k finite.
7. $\mathrm{Spec}(R) = V(1)^c$, so the whole space is "distinguished". It follows that $\mathrm{Spec}(R)$ is compact.

Easy direction of Hochster's Theorem

Spec(R)

1. is sober, because

   closed irreducible = $V(I)$, I a $\cap$-irred. radical ideal = $V(\mathfrak{p})$

2. is compact, because

   $\mathrm{Spec}(R) = D(1) = V(1)^c$

3. has a basis of compact open sets, because

   open = $V(I)^c$
   compact open = $V(\langle f_1, \ldots, f_k \rangle)^c$
   Every I is a join of $\langle f_1, \ldots, f_k \rangle$'s (in fact, of $\langle f \rangle$'s)

4. has the property that the intersection of two compact open sets is compact, because

   $V(I)^c \cap V(J)^c = (V(I) \cup V(J))^c = V(I \cap J)^c = V(IJ)^c$

   $(i_1, \ldots, i_k)(j_1, \ldots, j_\ell) = (\{i_r j_s\})$