Spec(R)

Short story long

The collection of radical ideals of a commutative ring R, ordered by inclusion, is a complete lattice, which satisfies an infinite distributive law.

1. $\bigvee r(J_k) := r\left(\sum r(J_k)\right)$.
2. $\bigwedge r(J_k) := \bigcap r(J_k)$.
3. $(\dagger)\ r(I) \wedge \bigvee r(J_k) = \bigvee (r(I) \wedge r(J_k))$.
(I.e. $r(I) \cap \left(r(\sum r(J_k))\right) = r\left(\sum r(I) \cap r(J_k)\right)$.)

This might trigger a memory from some topology class. If X is a topological space, then the lattice $\mathcal{O}(X)$ of open sets of X ordered by inclusion, and with the set-theoretic operations of union and intersection, is a complete lattice satisfying the same infinite distributive law $(\dagger)$: $O \cap \bigcup U_i = \bigcup (O \cap U_i)$.
A reasonable language for such structures is that of

1. top element $1 = X$,
2. bottom element $0 = \emptyset$,
3. finite meet $\wedge = \cap$, and
4. infinite join $\bigvee = \bigcup$.

Frames

A *frame* is a lattice in the language of $0, 1, \wedge, \bigvee$ that satisfies the infinite distributive law $(\dagger)$.

A typical example of a frame is $\mathcal{O}(X)$ for some topological space X.

Conversely, to each frame L there is a uniquely determined *sober* topological space such that $L \cong \mathcal{O}(X)$. Thus frames are complete algebraic invariants for sober spaces.

In fact, $X \mapsto \mathcal{O}(X)$ is the object part of a *contravariant* functor $\mathcal{O} : \mathbf{Top} \to \mathbf{Frame}$. from spaces to frames. On morphisms, $\mathcal{O}(f) = f^{-1}$: if $f : X \to Y$ is continuous, then

$$\mathcal{O}(f) : \mathcal{O}(Y) \to \mathcal{O}(X) : U \mapsto f^{-1}(U).$$

Recovering a space from its frame

Let X be a topological space, and p a point in X. We may identify p with a (continuous) map $p : \{\bullet\} \to X : \bullet \mapsto p$.

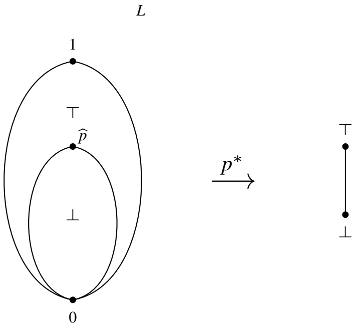
Apply $\mathcal{O}$ to this situation:

$$p^* : \mathcal{O}(X) \to \mathcal{O}(\{\bullet\}) = \{\emptyset, \{\bullet\}\} = \{\perp, \top\} = \{\text{\bot}, \text{\top}\}.$$

p^* is a frame homomorphism *onto* the 2-element frame. May think of p^* as a character of L, or as a characteristic function defined on L.
$p \in U \in \mathcal{O}(X)$ is equivalent to $p(\bullet) \in U$, hence to $p^{-1}(U) = \{\bullet\}$, which is equivalent to $p^*(U) = \top$.

This allows us to rebuild X from its frame of open sets. If L is a frame, then set of points of the associated space is the set $\mathbf{pt}(L) = \mathrm{Hom}(L, \mathcal{O}(\bullet))$ of characters $p^* : L \to \{\perp, \top\}$. A subset of points, $V \subseteq \mathbf{pt}(L)$, is open if it is represented by an element of L. That is, if there is an element $v \in L$ such that $V = \mathbf{pt}(v) = \{p^* \in \mathbf{pt}(L) : p^*(v) = \top\}$.

Characters correspond to meet-irreducible frame elements.



Commutative rings (Summary)

We have a closure operator

$$r : \mathrm{Idl}(R) \to L : I \mapsto r(I).$$

Claim. This is a $(\wedge, \bigvee)$-lattice homomorphism from $\mathrm{Idl}(R)$ onto a frame. (Check.)

The kernel of this map encodes the part of R that can be "understood via linear algebra".

The image frame L is the reflection of topological space, called the *spectrum* of the ring, $\mathrm{Spec}(R)$. The points of the spectrum are: frame characters $\rightsquigarrow$ proper meet-irreducible frame elements (= largest element in the pre-image of $\perp$) $\rightsquigarrow$ proper meet-irreducible radical ideals (= primes ideals).

Exercise. TFAE for $I \lhd R$.

1. I is prime.
2. I is meet-irreducible and semiprime ($r(I) = I$).

Commutative rings (Topology)

Let $\mathrm{Spec}(R) = \{\mathfrak{p} \lhd R \mid \mathfrak{p} \text{ is prime}\}$. For a subset $S \subseteq R$ let

$$V(S) := \{\mathfrak{p} \in \mathrm{Spec}(R) \mid S \subseteq \mathfrak{p}\}.$$

Take the sets $V(S)$ to be the closed sets of the topology on $\mathrm{Spec}(R)$.

The prime spectrum of a 1-element ring is empty. The prime spectrum of a field (a simple commutative ring) is a point.

Exercise.

1. Draw the ideal lattice of $R = \mathbb{Z}_{72} = \mathbb{Z}_{2^3 \cdot 3^2}$.
2. For each covering $I \prec J$, $J^2 \subseteq I$, determine the module structure on J/I.
3. Determine $\mathrm{Spec}(R)$ up to homeomorphism.