# Primary Decomposition: Lasker-Noether Theorem

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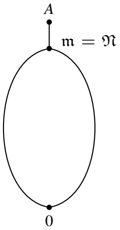


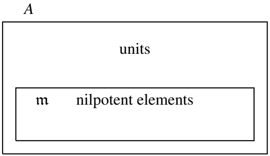

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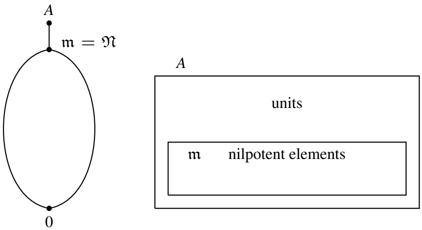


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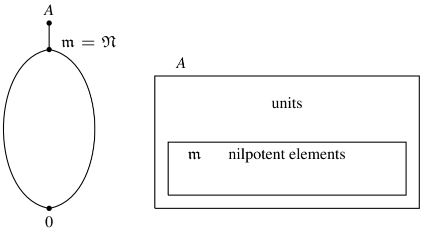


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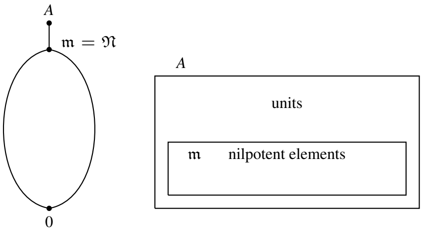


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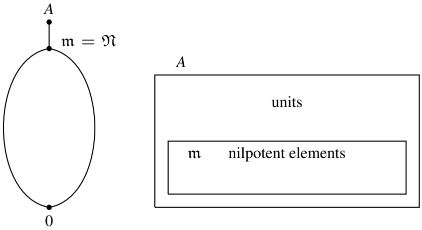


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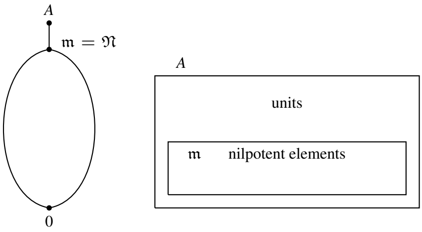


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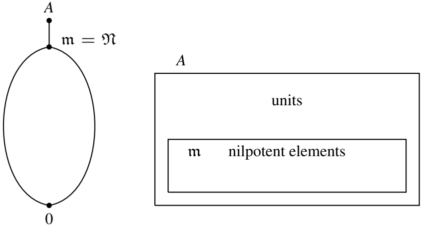


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(Second Uniqueness Theorem: AM, Theorem 4.10)

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**Krull Intersection Thm.** If $A$ is Noetherian, $I \lhd A$ and $K = \bigcap_{n<\omega} I^n$, then there exists $i \in I$ such that $(1-i)K = 0$. Hence if $I$ is a proper ideal, then $\bigcap_{n<\omega} I^n = (0)$ in the following cases:

(a) $I \subseteq J(A)$. (Mention Jacobson's Conjecture!)

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• Cyclic modules of the form $A/\mathfrak{p}$ behave like the more specialized modules $S = A/\mathfrak{m}$ ($\mathfrak{m}$ maximal), which are typical simple modules. It is not unreasonable to think of associated primes as specifying a kind of torsion in $M$: $\mathfrak{p} \in \mathrm{Ass}(M)$ iff $\mathfrak{p} = (0 : m)$ for some $m \in M$, so $m$ is an "exact $\mathfrak{p}$-torsion" element.

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**Example.** If $0 \to \mathbb{Z} \xrightarrow{\alpha} \mathbb{Z}_2 \times \mathbb{Z} \xrightarrow{\beta} \mathbb{Z}_2 \times \mathbb{Z}_3 \to 0$ is exact, where $\alpha(m) = (0, 3m)$, then $\mathrm{Ass}(L) = \{(0)\}$, $\mathrm{Ass}(M) = \{(0), (2)\}$, and $\mathrm{Ass}(N) = \{(2), (3)\}$. Thus none of the inclusions need be equality.

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Here we consider the effect on associated primes of restriction of scalars along $A \to S^{-1}A$. ($A \to S^{-1}A$ induces a continuous injection $\mathrm{Spec}(S^{-1}A) \to \mathrm{Spec}(A)$, which we treat as inclusion.)

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Conversely if $m \in M \setminus \{0\}$ and $\mathfrak{p} = (0 : m)_A \in \mathrm{Ass}_A(M)$, then $\mathfrak{p} \cap S = \emptyset$ (else $m = 0$), so $\mathfrak{p}(S^{-1}A)$ is prime. We claim that $\mathfrak{p}(S^{-1}A) = (0 : m)_{S^{-1}A}$, so that $\mathfrak{p}(S^{-1}A) \in \mathrm{Ass}_{S^{-1}A}(M)$. To see this, note that $r/s \in (0 : m)_{S^{-1}A}$ iff $\exists t \in S(trm = 0)$ iff $\exists t \in S(tr \in (0 : m)_A = \mathfrak{p})$ iff $r \in \mathfrak{p}$.

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- A finite intersection of $\mathfrak{p}$-primary submodules is $\mathfrak{p}$-primary.

# Primary Submodules

**Defn.** A submodule $N \lneq M$ is *primary* if for all $a \in A$ the map $\lambda_a \colon M/N \to M/N \colon \overline{m} \to a\overline{m}$ is injective or nilpotent.

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- the set $\mathfrak{p}$ of $a \in A$ where $\lambda_a$ is not injective is a prime ideal. ($N$ is $\mathfrak{p}$-*primary*.)
- A finite intersection of $\mathfrak{p}$-primary submodules is $\mathfrak{p}$-primary.
- If $N \lneq M$ is $\mathfrak{p}$-primary and $m \in M \setminus N$, then $(N : m)$ is a $\mathfrak{p}$-primary ideal.

# Connection Between Associated Primes and Primary Submodules, I

Assume that $A$ is Noetherian and $M$ is a f.g. $A$-module.

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Assume that $A$ is Noetherian and $M$ is a f.g. $A$-module.

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Assume that $A$ is Noetherian and $M$ is a f.g. $A$-module.

**Thm.** If $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$, then $N$ is $\mathfrak{p}$-primary.

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Assume that $A$ is Noetherian and $M$ is a f.g. $A$-module.

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*Proof.* If $\mathrm{Ass}(M/N) = \{\mathfrak{p}\}$, then $\bigcap \mathrm{Supp}(M/N) = \mathfrak{p}$.

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Assume that $A$ is Noetherian and $M$ is a f.g. $A$-module.

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# Primary Decomposition For Modules

**Thm.** If $Q \lneq M$ is a meet-irreducible $A$-submodule, then $\mathrm{Ass}(M/Q)$ has at most one element. If also $A$ is Noetherian and $M$ is f.g., then $\mathrm{Ass}(M/Q)$ has exactly one element, and $Q$ is primary.

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*Proof.* If $\operatorname{Ass}(M/Q) = \{\mathfrak{p}, \mathfrak{q}, \ldots\}$ has more than one element, then $M/Q$ has nonisomorphic submodules $N_1 \cong A/\mathfrak{p}$ and $N_2 \cong A/\mathfrak{q}$.

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**Thm.** If $N = Q_1 \cap \cdots \cap Q_k$ then $\mathrm{Ass}(M/N) \subseteq \bigcup \mathrm{Ass}(M/Q_i)$. When $A$ is Noetherian, equality holds provided $N = Q_1 \cap \cdots \cap Q_k$ is an irredundant representation and each $Q_i$ is primary. If $N = Q_1 \cap \cdots \cap Q_k$ is a minimal representation, then the associated primes of the factors are uniquely determined.