Nullstellensatz = Zero Locus Theorem

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An informal way to say this: $A \subseteq k^n$ is Galois-closed iff it is definable by polynomials, and $F \subseteq k[\overline{x}]$ is Galois-closed iff it is definable by points.

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The assertion that $p \in I(V(f))$ means $(\forall \overline{a})((f(\overline{a}) = 0) \to (p(\overline{a}) = 0))$. In this example, $V(I(f)) = \langle x^2 + y^2 - 1 \rangle$.

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The nonobvious claim of this theorem is that every radical ideal is Galois-closed.

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Proof. Induction on n.
True when $n = 1$, since $k[a_1]$ is not a field if a_1 is transcendental over k, and is a finite dimensional field extension of k if a_1 is algebraic over k.

Let $A = k[a_1]$ and let $k' = k(a_1) \leq K$. $K = k[a_1, \ldots, a_n] = k'[a_2, \ldots, a_n]$, so, by induction, K is a finite dimensional field extension of k'. Each a_i, $i > 1$, satisfies a monic polynomial $p_i(X)$ over $k' = k(a_1)$. If a_1 is algebraic over k, then $k \leq_{\text{fin}} k' \leq_{\text{fin}} K$, done. Assume that a_1 is transcendental over k. $A = k[a_1] \cong k[x]$ and $k' \cong k(x)$. Let $f \in A$ be the product of all denominators of coefficients of the $p_i(X)$ when considered as rational functions in a_1. Each $a_i, i > 1$, is integral over A_f. Since $A_f \leq_{\text{int}} A_f[a_2, \ldots, a_n] = K^{\text{field}}$, A_f is a field. This is impossible, since $A \cong k[x]$ has infinitely many primes and $f \neq 0$ cannot be divisible by all of them.

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