Modules, III

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In particular, if the underlying set functor $U : \mathcal{C} \to \mathbf{Set}$ is representable, then the underlying set of a product is the product of the underlying sets of the factors.

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We call $(M \oplus N, p_M, p_N, \iota_M, \iota_N)$ the biproduct of M and N.