Jacobson Radical, Nakayama's Lemma



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An element $r \in R$ belongs to the *Jacobson radical* of R if it acts nilpotently on all left R-modules of finite length. This is equivalent to saying that $rS = \{0\}$ whenever S is a simple R-module.

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The identity function on M is representable as an $n \times n$ matrix A with coefficients in I.

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