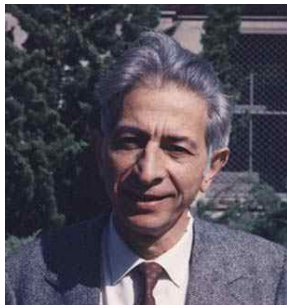


Integral Dependence



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is a chain of primes in B satisfying $\mathfrak{q}_i^c = \mathfrak{p}_i$ for $i = 1, \dots, m$ ($\leq n$), then the \mathfrak{q} -chain can be extended to a length- n chain ($\mathfrak{q}_i^c = \mathfrak{p}_i$ for $i = 1, \dots, n$).

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Cor. If $A \leq_{\text{int}} B$ and either has finite Krull dimension, then both have the same Krull dimension.

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Cor. If $A \leq_{\text{int}} B$ and either has finite Krull dimension, then both have the same Krull dimension. (So any number ring has Krull dimension 1.)

Going Down

Thm. Assume that $A \leq_{\text{int}} B$ are integral domains and A is integrally closed. If

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Proof. See AM Theorem 5.16, or Eisenbud Theorem 13.9.

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Df. A ring extension $A \leq B$ has the “going up” or “going down” property if chains or primes can be extended as appropriate.

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Proof. Assume that $K = k[a_1, \dots, a_t, \dots, a_n]$ where a_1, \dots, a_t are a transcendence base for K over k . Replace k by $k(a_1, \dots, a_{t-1})$ to reduce to the case $t = 1$.

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Lemma. Let K/k be an extension of fields. If K is finitely generated as a k -algebra, then K is finitely generated as a k -module. (That is, K an algebraic extension of k .)

Proof. Assume that $K = k[a_1, \dots, a_t, \dots, a_n]$ where a_1, \dots, a_t are a transcendence base for K over k . Replace k by $k(a_1, \dots, a_{t-1})$ to reduce to the case $t = 1$. If $A = k[a_1]$, then $\exists c \in k[a_1] - \{0\}$ such that all a 's are integral over $A[1/c]$,

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Cor. A finitely generated ring that is a field is a finite field.

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Cor. A finitely generated ring that is a field is a finite field. (Equivalently, every maximal ideal of $\mathbb{Z}[x_1, \dots, x_n]$ has finite index.)