

# Functoriality of Spec

# Functors

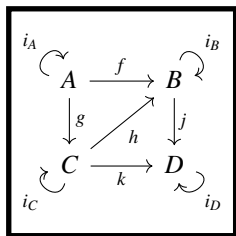
A functor is a homomorphism of categories.

$$F : \mathcal{C} \rightarrow \mathcal{D}.$$

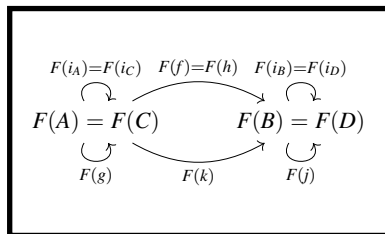
More informatively, a covariant functor is an “arrow-preserving homomorphism”, while a contravariant functor is an “arrow-reversing homomorphism”. Instead of writing about contravariant functors,  $F : \mathcal{C} \rightarrow \mathcal{D}$ , we may write about covariant functors  $G : \mathcal{C}^{\text{op}} \rightarrow \mathcal{D}$ .

Think:  $F$  “constructs” from each object of  $\mathcal{C}$  an object of  $\mathcal{D}$ , but it does so in a “uniform way”. The uniformity of the construction is indicated in that  $F$  is defined also on morphisms of  $\mathcal{C}$  and that  $F$  is a homomorphism of categories.

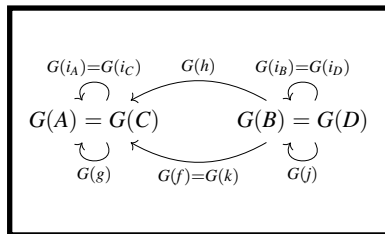
# Covariant $F$ versus contravariant $G$



$F$



$G$



# Claim

$\text{Spec} : \mathbf{CRng}^{\text{op}} \rightarrow \mathbf{Top}$  is a contravariant functor from the category of commutative rings to the category of topological spaces.

# Check the details

Given  $\alpha : R \rightarrow S$ , must assign continuous  $F(\alpha) : \operatorname{Spec}(S) \rightarrow \operatorname{Spec}(R)$  and then check that the assignment yields a contravariant functor.

**First.** Take  $F(\alpha) = \alpha^{-1}$ . Given  $I \triangleleft S$ , the “contraction” of  $I$  along  $\alpha$  is  $\alpha^{-1}(I) = \alpha^*(I)$ . That is, if  $I$  is the kernel of the natural map  $S \xrightarrow{\nu} S/I$ , then the contraction of  $I$  along  $\alpha$  is the kernel of the composite  $R \xrightarrow{\alpha} S \xrightarrow{\nu} S/I$ .

**Second.** The contraction of a prime is prime. If  $I$  is the kernel of a homomorphism of  $S$  into a field,  $S \xrightarrow{\nu} S/I \leq \mathbb{F}$ , then  $\alpha^{-1}(I)$  is the kernel of a homomorphism of  $R$  into a field,  $R \xrightarrow{\alpha} S \xrightarrow{\nu} S/I \leq \mathbb{F}$ . This shows that  $\alpha^* : \operatorname{Spec}(S) \rightarrow \operatorname{Spec}(R)$  is a function.

**Third.**  $\alpha^*$  is continuous. It suffices to prove that  $(\alpha^*)^{-1}(D(f))$  is open for any  $f \in R$ . But  $(\alpha^*)^{-1}(D(f)) = D(\alpha(f))$ .

$$\begin{aligned} \mathfrak{p} \in (\alpha^*)^{-1}(D(f)) &\Leftrightarrow \alpha^*(\mathfrak{p}) \in D(f) \Leftrightarrow f \notin \alpha^*(\mathfrak{p}) \\ &\Leftrightarrow \alpha(f) \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in D(\alpha(f)) \end{aligned}$$

# Spec is a contravariant functor

Must check:

①  $F(\alpha \circ \beta) = F(\beta) \circ F(\alpha).$   
 $(\alpha \circ \beta)^* = (\alpha \circ \beta)^{-1} = \beta^{-1} \circ \alpha^{-1} = \beta^* \circ \alpha^*.$

②  $F(\text{id}_R) (= \text{id}_R^*) = \text{id}_{\text{Spec}(R)}.$   
Contraction along the identity homomorphism is the identity map on (prime) ideals.

③  $F(\text{dom}(\alpha)) = \text{cod}(F(\alpha)).$   
If  $\alpha : R \rightarrow S$ , then  $F(\alpha) : \text{Spec}(S) \rightarrow \text{Spec}(R)$ , so  
 $F(\text{dom}(\alpha)) = \text{Spec}(R) = \text{cod}(F(\alpha)).$

④  $F(\text{cod}(\alpha)) = \text{dom}(F(\alpha)).$   
If  $\alpha : R \rightarrow S$ , then  $F(\alpha) : \text{Spec}(S) \rightarrow \text{Spec}(R)$ , so  
 $F(\text{cod}(\alpha)) = \text{Spec}(S) = \text{dom}(F(\alpha)).$

# Some properties of Spec

- ① If  $\nu : R \rightarrow R/I$  is the natural map, then  $\nu^* : \operatorname{Spec}(R/I) \rightarrow \operatorname{Spec}(R)$  is a topological embedding with image equal to the closed set  $V(I)$ .
- ②  $\operatorname{Spec}(S \times T) \cong \operatorname{Spec}(S) \sqcup \operatorname{Spec}(T)$ . (Disjoint union.)
- ③  $\operatorname{Spec}(R)$  is disconnected iff  $R \cong S \times T$  where  $|S|, |T| > 1$ . ( $R$  is directly decomposable.)

Proof of Item (3) ( $\Rightarrow$ ):

Assume  $\operatorname{Spec}(R) = U \sqcup U^c$  is a separation. ( $U$  is nonempty, proper, clopen.)

Then  $U = V(I)$ ,  $U^c = V(J)$  for some  $I, J \triangleleft R$ .

$V(I + J) = V(I) \cap V(J) = \emptyset$ , so  $I + J = R$ .

$V(I \cap J) = V(I) \cup V(J) = \operatorname{Spec}(R)$ , so  $I \cap J \subseteq \mathfrak{N}$ .

$\bar{I} = I/(I \cap J)$  and  $\bar{J} = J/(I \cap J)$  are complementary in  $R/(I \cap J)$ , so they are generated by complementary idempotents:  $\bar{I} = (\bar{e})$ ,  $\bar{J} = (\bar{I} - \bar{e})$ .

By HW 5(b), the fact that  $I \cap J \subseteq \mathfrak{N}$  implies  $R$  has an idempotent  $e$  such that  $e/(I \cap J) = \bar{e}$ . Necessarily  $e \in I$ ,  $U = V(e)$ ,  $U^c = V(1 - e)$ , and  $R \cong R/(e) \times R/(1 - e)$ .