Functoriality of Spec

Functors

A functor is a homomorphism of categories.

$$F : \mathcal{C} \to \mathcal{D}.$$

More informatively, a covariant functor is an "arrow-preserving homomorphism", while a contravariant functor is an "arrow-reversing homomorphism". Instead of writing about contravariant functors, $F : \mathcal{C} \to \mathcal{D}$. we may write about covariant functors $G : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Think: F "constructs" from each object of $\mathcal{C}$ an object of $\mathcal{D}$, but it does so in a "uniform way". The uniformity of the construction is indicated in that F is defined also on morphisms of $\mathcal{C}$ and that F is a homomorphism of categories.

Covariant F versus contravariant G

$$
\begin{tikzcd}
A \arrow["i_A", loop above] \arrow[r, "f"] \arrow[d, "g"] & B \arrow["i_B", loop above] \arrow[d, "j"] \\
C \arrow["i_C", loop below] \arrow[ur, "h"'] \arrow[r, "k"'] & D \arrow["i_D", loop below]
\end{tikzcd}
$$

$\xrightarrow{F}$

$$
\begin{tikzcd}
F(A) = F(C) \arrow["F(i_A)=F(i_C)", loop above] \arrow["F(g)"', loop below] \arrow[r, bend left, "F(f)=F(h)"] \arrow[r, bend right, "F(k)"'] & F(B) = F(D) \arrow["F(i_B)=F(i_D)", loop above] \arrow["F(j)"', loop below]
\end{tikzcd}
$$

$\xrightarrow{G}$

$$
\begin{tikzcd}
G(A) = G(C) \arrow["G(i_A)=G(i_C)", loop above] \arrow["G(g)"', loop below] & G(B) = G(D) \arrow["G(i_B)=G(i_D)", loop above] \arrow["G(j)"', loop below] \arrow[l, bend right, "G(h)"'] \arrow[l, bend left, "G(f)=G(k)"]
\end{tikzcd}
$$

Claim

Spec : $\mathbf{CRng}^{\mathrm{op}} \to \mathbf{Top}$ is a contravariant functor from the category of commutative rings to the category of topological spaces.

Check the details

Given $\alpha : R \to S$, must assign continuous $F(\alpha) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ and then check that the assignment yields a contravariant functor.

First. Take $F(\alpha) = \alpha^{-1}$. Given $I \lhd S$, the "contraction" of I along α is $\alpha^{-1}(I) = \alpha^*(I)$. That is, if I is the kernel of the natural map $S \stackrel{\nu}{\to} S/I$, then the contraction of I along α is the kernel of the composite $R \stackrel{\alpha}{\to} S \stackrel{\nu}{\to} S/I$.

Second. The contraction of a prime is prime. If I is the kernel of a homomorphism of S into a field, $S \stackrel{\nu}{\to} S/I \leq \mathbb{F}$, then $\alpha^{-1}(I)$ is the kernel of a homomorphism of R into a field, $R \stackrel{\alpha}{\to} S \stackrel{\nu}{\to} S/I \leq \mathbb{F}$. This shows that $\alpha^* : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$ is a function.

Third. α^* is continuous. It suffices to prove that $(\alpha^*)^{-1}(D(f))$ is open for any $f \in R$. But $(\alpha^*)^{-1}(D(f)) = D(\alpha(f))$.

$$\begin{aligned} \mathfrak{p} \in (\alpha^*)^{-1}(D(f)) \quad &\Leftrightarrow \alpha^*(\mathfrak{p}) \in D(f) \Leftrightarrow f \notin \alpha^*(\mathfrak{p}) \\ &\Leftrightarrow \alpha(f) \notin \mathfrak{p} \Leftrightarrow \mathfrak{p} \in D(\alpha(f)) \end{aligned}$$

Spec is a contravariant functor

Must check:

1. $F(\alpha \circ \beta) = F(\beta) \circ F(\alpha)$.
   $(\alpha \circ \beta)^* = (\alpha \circ \beta)^{-1} = \beta^{-1} \circ \alpha^{-1} = \beta^* \circ \alpha^*$.

2. $F(\mathrm{id}_R) \;\; (= \mathrm{id}_R^*) \quad = \mathrm{id}_{\mathrm{Spec}(R)}$.
   Contraction along the identity homomorphism is the identity map on (prime) ideals.

3. $F(\mathrm{dom}(\alpha)) = \mathrm{cod}(F(\alpha))$.
   If $\alpha : R \to S$, then $F(\alpha) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$, so
   $F(\mathrm{dom}(\alpha)) = \mathrm{Spec}(R) = \mathrm{cod}(F(\alpha))$.

4. $F(\mathrm{cod}(\alpha)) = \mathrm{dom}(F(\alpha))$.
   If $\alpha : R \to S$, then $F(\alpha) : \mathrm{Spec}(S) \to \mathrm{Spec}(R)$, so
   $F(\mathrm{cod}(\alpha)) = \mathrm{Spec}(S) = \mathrm{dom}(F(\alpha))$.

Some properties of Spec

1. If $\nu : R \to R/I$ is the natural map, then $\nu^* : \mathrm{Spec}(R/I) \to \mathrm{Spec}(R)$ is a topological embedding with image equal to the closed set $V(I)$.
2. $\mathrm{Spec}(S \times T) \cong \mathrm{Spec}(S) \sqcup \mathrm{Spec}(T)$. (Disjoint union.)
3. $\mathrm{Spec}(R)$ is disconnected iff $R \cong S \times T$ where $|S|, |T| > 1$. (R is directly decomposable.)

Proof of Item (3) ($\Rightarrow$):
Assume $\mathrm{Spec}(R) = U \sqcup U^c$ is a separation. (U is nonempty, proper, clopen.)
Then $U = V(I), U^c = V(J)$ for some $I, J \lhd R$.
$V(I + J) = V(I) \cap V(J) = \emptyset$, so $I + J = R$.
$V(I \cap J) = V(I) \cup V(J) = \mathrm{Spec}(R)$, so $I \cap J \subseteq \mathfrak{N}$.
$\overline{I} = I/(I \cap J)$ and $\overline{J} = J/(I \cap J)$ are complementary in $R/(I \cap J)$, so they are generated by complementary idempotents: $\overline{I} = (\overline{e}), \overline{J} = (\overline{1} - \overline{e})$.
By HW 5(b), the fact that $I \cap J \subseteq \mathfrak{N}$ implies R has an idempotent e such that $e/(I \cap J) = \overline{e}$. Necessarily $e \in I$, $U = V(e), U^c = V(1 - e)$, and $R \cong R/(e) \times R/(1 - e)$.