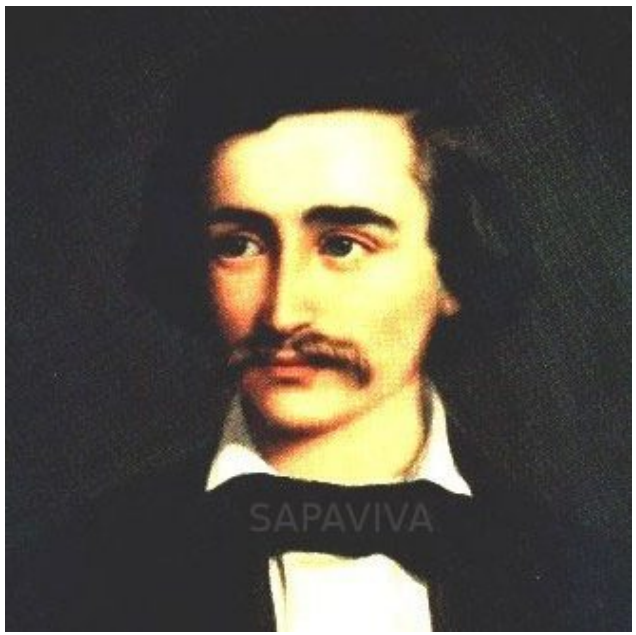
Dedekind domains



A case of Fermat's Last Theorem

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The fact that $x^p - 1$ factors as $(x-1)(x-\omega)\cdots(x-\omega^{p-1})$ for $\omega = e^{\frac{2\pi i}{p}}$ implies that

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This forces $(x+\omega y) = u\alpha^p$ for some $u, \alpha \in \mathbb{Z}[\omega]$, u a unit.

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Conclusion. There are no Case 1 solutions to $x^p + y^p = z^p$ if $p \geq 5$ and $\mathbb{Z}[\omega_p]$ is a UFD.

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3. (If time) Discuss the ideal class group.

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Cor. Additively, $\mathcal{O}_K$ is free of rank n.