The local structure of a Dedekind domain

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PID/PIR	yes	no, $(\mathbb{F}_2)^{\omega}$
Noetherian	yes	no, $(\mathbb{F}_2)^{\omega}$
Artinian	yes	no, $(\mathbb{F}_2)^{\omega}$
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$\dim \leq n$	yes	yes, $(A^{\text{domain}} = \cap_{\mathfrak{m}} A_{\mathfrak{m}})$
$I + J = K$	yes	yes
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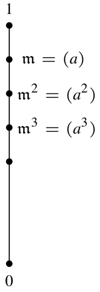

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Cor. A Noetherian domain D is Dedekind iff $D_{\mathfrak{m}}$ is a DVR for all maximal $\mathfrak{m} \lhd D$.

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